**Phase Velocity vs. Group Velocity**

We should perhaps be (at least) a little bit disturbed that the real part of the index of refraction can be less than 1 for some frequencies. When we're working with just one plane wave, that wave has a single, definite frequency (ω) and wavelength ($\lambda = 2\pi/k$), and the "phase velocity" (the speed at which that single plane wave travels in a medium with index of refraction $\tilde{n} = n_R + in_I$) is given by:

$$v_p = \frac{\omega}{k_R} = \frac{c}{n_R}$$

For $n_R < 1$, this implies that the phase velocity is greater than c! This seems to violate special relativity, so does that mean that our formula is incorrect? No, it means we have to remember that plane waves are mathematical idealizations, and we need to be <u>very</u> careful when trying to physically interpret the behavior of something that is <u>very</u> unphysical.

Any real signal is going to be made from a sum of many different frequencies, which travel together as a group, at a speed that will always be less than or equal to the speed of light in vacuum. We study plane waves because, if we understand what a single plane wave does, then we can also analyze how a linear sum of plane waves behaves.

To illustrate this, consider a plane wave propagating in the $+z$-direction, given by the equation $f(t) = A\exp[i(kz - \omega t)]$. This plane wave exists at all points in space ($-\infty < z < +\infty$), and not just along the z-axis but everywhere in x and y as well. It also exists at all times ($-\infty < t < +\infty$), from forever in the past to forever in the future. Not very physical at all! If we're looking at just one point in space, the function $f(t) = A\exp(-i\omega t)$ will look like:

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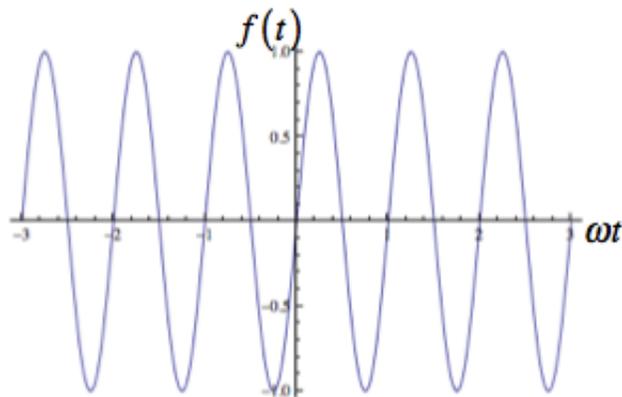
As a (slightly) more realistic example, suppose we start transmitting from some point in space just a single frequency at $t = 0$. This function $f(t > 0) = A\exp(-i\omega t)$ (again, suppressing the spatial dependence) is going to look like:
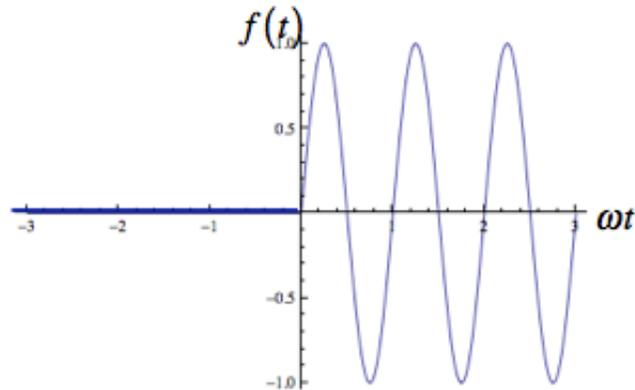
This is not a plane wave!! It doesn't meet our requirement of extending over all space and time. However, it can be broken down into a sum of plane waves (which is actually true for any well-behaved function, even if it's not periodic). Fourier decomposition tells us exactly what the distribution of frequencies is for a given function.

To see how, let's start by considering only the time-dependent part of our plane waves, which is equivalent to looking at how the function varies at a fixed point in space, and then later consider what happens when we include the spatial dependence, and also the frequency dependence of the index of refraction.

We can represent a time-dependent function $f(t)$ as an infinite sum of plane waves over a continuous range of frequencies (ω) by writing the sum as an integral:

$$f(t) = \int_{-\infty}^{+\infty} A(\omega)\exp(-i\omega t)\, d\omega$$

The various values of the coefficients $A(\omega)$ tell us how much of each frequency contributes to the total function. We can invert this expression to find the coefficients if we remember that:

$$\int_{-\infty}^{+\infty} \exp(-i\omega t)\, dt = 2\pi \cdot \delta(\omega)$$

where $\delta(\omega)$ is the Dirac delta function. [See Griffiths, p. 46] This should make sense, since the cosines and sines average out to zero when we integrate over all times if the frequency is non-zero ($\omega \neq 0$), and becomes infinite if the argument of the exponential equals zero ($\exp(0) = 1$). This is exactly the behavior of the delta

function; the factor of 2π just tells us about how quickly the function goes from being zero to becoming infinite.

If we multiply both sides of our equation for $f(t)$ by $\exp(i\omega' t)$, and then integrate both sides over time, we get:

$$\int_{-\infty}^{+\infty} f(t)\exp(i\omega' t)\, dt = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} A(\omega)\exp(-i\omega t)\exp(i\omega' t)\, d\omega\right] dt$$

But we can switch the order of integration, so that the right-hand side becomes:

$$\int_{-\infty}^{+\infty} A(\omega)\left[\int_{-\infty}^{+\infty}\exp(-i\omega t)\exp(i\omega' t)\, dt\right] d\omega$$

$$= \int_{-\infty}^{+\infty} A(\omega)\left[\int_{-\infty}^{+\infty}\exp\left[-i(\omega-\omega')t\right] dt\right] d\omega$$

$$= \int_{-\infty}^{+\infty} A(\omega)\left[2\pi\delta(\omega-\omega')\right] d\omega = 2\pi A(\omega')$$

The delta function just picks out the single frequency $\omega = \omega'$. So, the coefficients $A(\omega)$ are given by:

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\exp(i\omega t)\, dt$$

There are different conventions about where to stick the factor of $1/2\pi$, so be careful when reading other books.

Just to check our result, if our function $f(t)$ is a plane wave, $f(t) = A_0\exp(-i\omega t)$:

$$A(\omega') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A_0\exp(-i\omega t)\exp(i\omega' t)\, dt = A_0\delta(\omega-\omega')$$

Putting this back into our equation for the time-dependent function we started with:

$$f(t) = \int_{-\infty}^{+\infty}\left[A_0\delta(\omega-\omega')\right]\exp(-i\omega' t)\, d\omega' = A_0\exp(-i\omega t)$$

Just as we expected, the Fourier decomposition tells us that a plane wave is made up from a single frequency $\omega' = \omega$.

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Now, let's put in the spatial dependence. If our function depends on both space and time (let's restrict it to a function of z and t only), we can always fix a point in space and analyze how the function varies with time, or fix a point in time and analyze how the function varies in space; which is equivalent to doing a double integral over space and time:

$$A(k,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z,t) \exp(i\omega t) dt \right] \exp(-ikz) dz$$

$$\rightarrow \quad A(k,\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dz \, f(z,t) \exp\left[-i(kz - \omega t)\right]$$

$$\rightarrow \quad f(z,t) = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \, A(k,\omega) \exp\left[i(kz - \omega t)\right]$$

To get a feel for what's going on when a real signal travels through a dispersive medium, let's start with the simplest case, and see what happens when we add just two plane waves, each with equal amplitude. We're going to first need some trig identities to do this, but these can be easily gotten using complex exponentials.

If we multiply two complex exponentials, we get:

$$\exp(i\alpha)\exp(i\beta) = \exp\left[i(\alpha + \beta)\right] = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

We can also multiply these out to get:

$$\exp(i\alpha)\exp(i\beta) = \left[\cos(\alpha) + i\sin(\alpha)\right]\left[\cos(\beta) + i\sin(\beta)\right]$$

$$= \left[\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\right] + i\left[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\right]$$

Equating the real and imaginary parts leads us to:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Using this result, if we add $\cos(\alpha + \beta)$ and $\cos(\alpha - \beta)$, we get:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$$

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Now let $a = \alpha + \beta$ & $b = \alpha - \beta$, then solving for α & β in terms of a & b, we have the result we're looking for:

$$\alpha = \frac{a+b}{2} \qquad \& \qquad \beta = \frac{a-b}{2}$$

$$\Rightarrow \quad \cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Now that we have this identity, suppose we're adding two plane waves with unit amplitude:

$$f_1(z,t) = \cos(k_1 z - \omega_1 t) \qquad \& \qquad f_2(z,t) = \cos(k_2 z - \omega_2 t)$$

Lets work with an ω_1 and ω_2 that are relatively close to each other, and also for the k's, because that's similar to when we're adding up a continuum of ω's and k's.

Define the following quantities:

$$\overline{k} \equiv \frac{k_1 + k_2}{2} \qquad \& \qquad \overline{\omega} \equiv \frac{\omega_1 + \omega_2}{2}$$

Notice that $\overline{k}$ and $\overline{\omega}$ are just the average wave number and the average frequency, respectively.

We'll also need:

$$\frac{\Delta k}{2} \equiv \frac{k_1 - k_2}{2} \qquad \& \qquad \frac{\Delta\omega}{2} \equiv \frac{\omega_1 - \omega_2}{2}$$

Δk and $\Delta\omega$ are just the difference between the wave numbers and the frequencies for the two waves. Using our final trig identity, the sum of the two waves is:

$$f_1(z,t) + f_2(z,t) = \cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t)$$

$$= 2\cos\left(\frac{\Delta k}{2} z - \frac{\Delta\omega}{2} t\right)\cos\left(\overline{k} \cdot z - \overline{\omega} \cdot t\right)$$

$$\equiv A(z,t)\cos\left(\overline{k} \cdot z - \overline{\omega} \cdot t\right)$$

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The result is a fast oscillating wave that's traveling with the "average wave number" ($\bar{k}$) and the "average frequency" ($\bar{\omega}$), with a phase velocity:

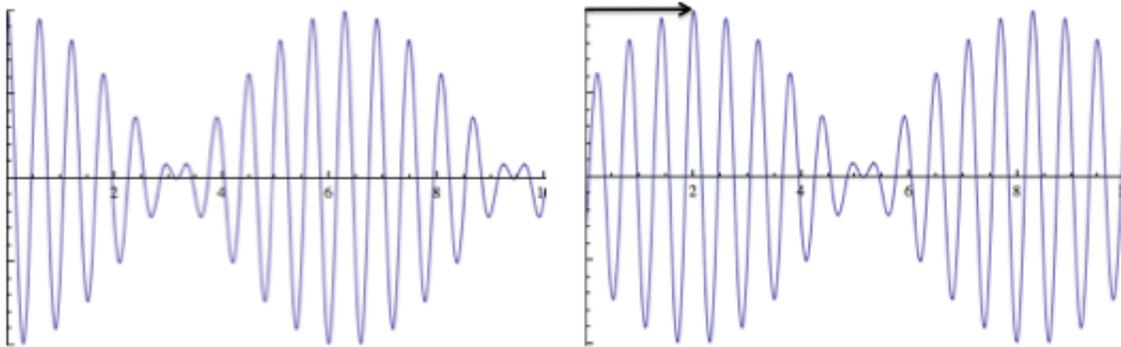$$v_p = \frac{\bar{\omega}}{\bar{k}}$$

But the amplitude of this wave is being modulated in space and time by:

$$A(z,t) = 2\cos\left(\frac{\Delta k}{2}z - \frac{\Delta\omega}{2}t\right)$$

This modulation travels at the "group velocity":

$$v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}$$

We can see that the group velocity is the wave speed of the packet in the figure below, which shows the sum of two waves at $t = 0$ (left side) and at a short time later ($t = 2$, right side):

Even though the phase advances at the phase velocity, the total wave packet (the envelope that contains the fast oscillation – i.e., the "signal") travels at the group velocity. It may take a little thought to convince yourself of this; it might help to look at the excellent "dispersion applet" at: http://www.falstad.com/dispersion/

If the two frequencies and wave numbers are very close to each other, as is the case when we're adding a continuous range of $k's$ and $\omega's$, this goes over to:

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega=\bar{\omega}}$$

The derivative is evaluated at the "central frequency" $\bar{\omega}$, essentially the "average" frequency of the wave packet.

Now, if the two waves are EM waves traveling in vacuum, then:

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = c \qquad \rightarrow \qquad v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{c(k_1 + k_2)}{k_1 + k_2} = c$$

But we also have:

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{c(k_1 - k_2)}{k_1 - k_2} = c$$

In this case, when the two wave speeds are the same, the group velocity is equal to the phase velocity!

What about when the wave speeds are not the same, like when EM waves are traveling in a dispersive medium, where the index of refraction is a function of frequency? Recall that our model for the index of refraction gave the relationship:

$$n = 1 + \frac{\alpha}{\omega_0^2 - \omega^2} \qquad \text{where} \quad \alpha \equiv \frac{Nq^2}{2m\varepsilon_0}$$

Let's not worry about the radiation damping right now, because the damping constant γ also has a frequency dependence, which would only make what follows all the more complicated. We want to show that the group velocity in a dispersive medium is going to be less than the speed of light in vacuum.

The wave number in the dispersive medium is:

$$k = \frac{\omega n}{c} = \frac{1}{c}\left(\omega + \frac{\alpha\omega}{\omega_0^2 - \omega^2}\right)$$

We can easily calculate $dk/d\omega$, and then invert it to find $d\omega/dk = 1/(dk/d\omega)$.

$$\frac{dk}{d\omega} = \frac{1}{c}\left(1 + \alpha\left[\frac{(\omega_0^2 - \omega^2) - \omega(-2\omega)}{(\omega_0^2 - \omega^2)^2}\right]\right)$$

$$= \frac{1}{c}\left(1 + \alpha\left[\frac{\omega_0^2 + \omega^2}{(\omega_0^2 - \omega^2)^2}\right]\right)$$

$$\Rightarrow \quad v_g = \left.\frac{d\omega}{dk}\right|_{\omega=\bar{\omega}} = \frac{c}{1+\dfrac{\alpha\left(\omega_0^2 + \bar{\omega}^2\right)}{\left(\omega_0^2 - \bar{\omega}^2\right)^2}} \quad < \quad c$$

Even when the phase velocity can be greater than c, the speed at which information travels (the group velocity) is less than c! We might be tempted to conclude that $v_g \to 0$ for $\bar{\omega} \to \omega_0$, but remember it was exactly that kind of behavior in the index of refraction (when it became infinite at $\omega = \omega_0$) that caused us to add in the radiation damping. Doing so makes the result more complicated, but the derivation above demonstrates the principle.

Now, we haven't told you the entire story, because in reality there are "fast-light" materials where we can not only have $n < 1$, the index of refraction can even take on negative values! This complicates matters, but I can assure you that that even in these kinds of materials, the time it takes for a signal to travel through the medium is still less than the time it takes the signal to travel through vacuum. There's a nice discussion of this (though somewhat challenging, at this level) in the short article:

"The speed of information in a fast-light optical medium"
Michael D. Stenner, Daniel J. Gauthier and Mark A. Neifeld
Nature, Vol. 425, p. 695-698 (October, 2003)