4.1 JG, ZR, VT, KJ

Vector Space - A set of vectors ~~on~~ which define the basis on which addition and multiplication by vectors & scalars can take place.

Subspace - a vector space which is a subset of a larger vector space.

What are the constraints of a subspace?

a line that does not pass through origin is not in vector space.

4.1 SD, BB, EV

A VECTOR SPACE IS A SET OF VECTORS THAT IS CLOSED UNDER VECTOR ADDITION AND SCALAR MULTIPLICATION.

A SUBSPACE IS A SUBSET OF A VECTOR SPACE WHICH IS CLOSED UNDER VECTOR ADDITION AND SCALAR MULTIPLICATION.

THE SPAN OF A SET OF VECTORS IS A SUBSPACE.

PRACTICE PROBLEMS

EXERCISES 5-8

4.2 JG, ZR, VT, KJ

The null space is a set of vectors that are squished onto the zero vector.

Nul A is the solution for all the x's to $A\vec{x} = \vec{0}$

Is ~~Nul A~~ the span of Nul A the span of the zero vector?

The column space is the explicit definition of all of the linear combinations of the columns of $m \times n$ matrix A

⇒ Also span of matrix A

~~⇒ Also~~

Null space is the Implicit definition of the linear combination of A

why is it called the Kernal?

Sec. 4.2 EV, BB, SD

Null space is a subspace of $\mathbb{R}^n$, and subspaces arise as a set of all solutions to $A\vec{x} = \vec{0}$.

COLUMN SPACE IS THE SET OF ALL LINEAR COMBINATIONS OF THE COLUMNS OF A MATRIX, AND IS A SUBSPACE OF $\mathbb{R}^m$.

A LINEAR TRANSFORMATION ON A VECTOR SPACE IS A FUNCTION ON VECTORS $\vec{x}$ WHICH IS DISTRIBUTIVE AND $T(c\vec{u}) = cT(\vec{u})$

4.2) Nul A

TS, IB, JR

- Null Space of an $m \times n$ matrix A; written as Nul A, is the set of all Solutions of the homogenous equation $Ax = 0$
- Null Space is also a Subspace of $\mathbb{R}^n$
- Null Space = Kernal

Col A

- For $A = [a_1 \dots a_n]$, Col A = Span$\{a_1, \dots, a_n\}$
- Column Space is Subspace $\mathbb{R}^m$

~~- Column is~~

- Col A is the range of A

SECTION 4.2

SM, SS, EL, KD

→ Null Space: A subspace of $\mathbb{R}^n$. It is the set of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$.

→ Trouble connecting Fig 1 to Theorem 2.

→ Column Space: (Col A) Set of all linear combinations ? of the columns of A.

· The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

→ Linear Transformations.

4.3 JG, KJ, VT, ZR

sets of Vectors

~~things~~ are linearly independent if they have access to R^n and are not a linear combination of the things that make them up.

sets are dependent if there is a solution (not a trivial one) to $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots = 0$

The basis is ~~a vector subspace~~ ~~of~~ a linearly independent set that spans the entire vector space

How do we describe vector space V now that we have these definitions? what is it used for ~~now~~? Is it useful for transformations?

Spanning set theorem:

If a vector is a linear combination of others, it's extra and we can get rid of it. we only need enough vectors to describe the space, and spans the space.

sidenote: what exactly is V?

4.3 BB, SD, EV

A BASIS OF A SUBSPACE IS A SET OF VECTORS SUCH THAT:

$\mathcal{B}$ IS AS LARGE AS POSSIBLE WHILE LINEARLY INDEPENDENT

$\mathcal{B}$ IS AS SMALL AS POSSIBLE WHILE SPANNING H.

THE PIVOT COLUMNS OF A ARE A BASIS FOR COL A.

EXERCISES 9-10, 13-14

IS, LS, JR

Basis| Let H be a subspace of vector space V. An indexed set of vectors $\mathcal{B} = \{b_1 \dots b_p\}$ in V is a basis for H iff

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) $H = \text{span}\{b_1 \dots b_p\}$.

(Basis is the smallest set of vectors that can make the whole space.)

Thrm. 5| You can find a basis in any spanning set of vectors.

Facts|

Basis for col A is ~~at~~ set of Pivot columns of A.

Basis for Nul A is the same size as the set of free variables in A.

Exercises|

11, 15, 16, 25, 28

SECTION 4.3 SM, SS, EL, KD

→ Basis Vectors:

- linearly independent set
- Define the span

→ Definition (pg 211): "Let H be a subspace of a vector space V." Why is this important?

- Pivot columns of a matrix A form a basis for Col A.

4.4 COORDINATE SYSTEMS

JG, KJ, VT, ZR

$\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$

A COORDINATE SYSTEM EXISTS IF UNIQUE SET OF SCALAR WEIGHTS c WILL SOLVE X WHEN APPLIED TO BASIS VECTORS

IS THE CURLY B ($\mathcal{B}$) A UNIT ~~[illegible]~~ VECTOR MATRIX?

THE COORDINATE VECTOR OF $\vec{x}$ IS THE SCALAR WEIGHTS APPLIED TO THE BASIS $\mathcal{B}$ THAT DEFINE $\vec{x}$

THE CHANGE OF COORDINATE MATRIX TRANSFORMS VECTOR FROM ONE BASIS TO ANOTHER

$\vec{x} = P_B[\vec{x}]_{\mathcal{B}}$

HOW IS THIS DIFFERENT FROM T(x) = TRANSFORMATIONS

ISOMORPHISM MAP SAME INFO TO NEW VECTOR SPACE.

ONE TO ONE TRANSFORMATION FROM ONE VECTOR SPACE TO ANOTHER.

4.4

SD, BB, EV

$[\vec{x}]_B$ IS THE VECTOR $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ THAT ~~[illegible]~~ REPRESENTS $\vec{x}$ IN THE BASIS $\mathcal{B}$.

THIS MAPPING $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ IS AN ISOMORPHISM FROM V TO $\mathbb{R}^n$.

4.4

TS, IB, JR

Coordinate Systems

- $\mathcal{B} = \{b_1, \dots b_n\}$ is a basis for V an x is in V.

The coordinates of x relative to the Basis $\mathcal{B}$ are the weights $c_1, \dots c_n$ such that $X = c_1 b_1 + \dots$ C

- Linear Transformations translate across coordinate Systems
- $A^{-1}MA$ gives you the result of a linear transform through a new coordinate system

Exercises

1, 3, 5, 7, 13, 25

Section 4.4

SM, SS, EL, KD

→ Coordinate system

Any vector $\vec{x}$ in a vector space V has a unique linear combination of the vector basis of that vector space V

→ Coordinate Mapping

Having basis vectors for a vector space V allows one to transform any vector from V to $\mathbb{R}^n$ and visa versa.

4.5 JG, KJ, VT, ZR

Theorem 9 - If there is a set number of $\mathcal{B}$ in a vector space V, and n vectors such that $n > \mathcal{B}$, some ~~vector~~ vectors must be linearly dependent.

The dimensions of $\mathcal{V}$ are the number of vectors that make up the basis of $\mathcal{V}$.

If H is a subspace of a finite space then then linearly independent columns of H can be expanded to the basis of H and
$\dim H \leq \dim V$

Any linearly independent set of elements in V can be used as a basis for V

dim Nul A $\Rightarrow$ # of free variables in $A\vec{x} = \vec{0}$

dim Col A $\Rightarrow$ # of pivot columns in A

what is a non finite set for which V is infinite dim?

4.5 BB, SD, EV

THE DIMENSION OF A VECTOR SPACE IS THE NUMBER OF VECTORS IN ITS BASIS.

THE DIMENSION OF A SUBSPACE IS ALWAYS LESS THAN OR EQUAL TO THE DIMENSION OF ITS PARENT SPACE.

DIM NUL $\underline{A}$ IS THE NUMBER OF FREE VARIABLES IN $\underline{A}\vec{x} = \vec{0}$.

DIM COL $\underline{A}$ IS THE NUMBER OF PIVOT COLUMNS IN $\underline{A}$.

4.5) The Dimension of a Vector Space TS, EB, JK

- A Vector Space V has basis of n Vectors, then every basis of V must consist of exactly n Vectors.
- dimension V, written as dim V, is the number of vectors in a basis of V
- $H \subseteq V$ then $\dim H \leq \dim V$
- Any linearly independant set of dim V elements in V is automatically a basis for V.

~~Exam~~ Exercises

21, 11, 5

Section 4.5 SM, SS, EL, KD

Dimensions of a vector space

- Basis vectors in a space are linearly independent
- Do basis vectors define a subset of space?
  - ↳ Basis vectors define a space!
- if H is a subspace of vector space V dimension of H $\leq$ dimension of V (a plane fits in a 3D space)
- Dimension of Nul A is defined by the number of free variables in $A\vec{x} = 0$ p 23
- Dimension of Col A is the number of pivot columns in A (in echelon form!)

4.6) JG, ER, VT, KJ

Rank $\Rightarrow$ dim col A
helps describe ~~the~~ thing in relation to the space it can occupy

~~does~~ $\mathbb{R}^n$ mean rank n?
row equivalent matrices have same column space.
# of pivot in A $\doteq$ Rank of A
Rank theorem: rank A + dim nul A = n

The row space is perpendicular to the column space, which is perpendicular to A^T null space.

Row A^T = col A

check out that col rank →

4.6 SD, BB, EV

ROW SPACE IS THE SET OF ALL LINEAR COMBINATIONS OF THE ROWS OF A MATRIX.

RANK OF A MATRIX IS DIM COL $\underline{A}$.
DIM COL $\underline{A}$ = DIM ROW $\underline{A}$.
RANK $\underline{A}$ + DIM NUL $\underline{A}$ = NUMBER OF COLUMNS IN $\underline{A}$.

EXERCISES 1-4, 5-16, 25

row space of A [redacted] space spanned by rows of A.

Dim row space A = Dim Col A = rank of A

continuation of IMT for n×n matrix A:
also equivalent
m. columns of A form basis for $\mathbb{R}^n$
n. Col A = $\mathbb{R}^n$
o. dim Col A = n
p. rank A = n
q. Nul A = {0}
r. dim Nul A = 0

4.6) TS, IB, JR

Section 4.6 SM, SS, EL, KD

Rank

— Rank of a matrix$^{\to m\times n}$ is the dimension of the column space ($\mathbb{R}^m$)

— What are the differences between column space and row space?

— If A is invertible
- Columns of A forms basis or $\mathbb{R}^n$ (because columns are linearly independent)
- Column space of A is $\mathbb{R}^n$
- Dimensions of Col A = n
- Rank A = n
- Nul A = {0} (you cant unsquish a line if youve gone from $\mathbb{R}^2 \to \mathbb{R}^1$)

4.7 Change of Bases

TS, IB, JR

$B = \{b_1, \dots b_n\}$ and $C = \{c_1, \dots c_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{C \leftarrow B}{P}$ such That

$$[X]_C = \underset{C \leftarrow B}{P} [X]_B$$

The columns of $\underset{C \leftarrow B}{P}$ are the coordinate vectors in the basis B

$$\underset{C \leftarrow B}{P} = \left[[b_1]_C \ [b_2]_C \cdots [b_n]_C \right]$$

$$\left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$$

What is the blue box Saying at the bottom of page 243.

20 a, 15, 14, 2

SECTION 4.7

SM, SS, EL, KD

→ Change of Basis: $[x]_C = \underset{C \leftarrow B}{P} [x]_B$

$\left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$ ← Notation: why do they point to the left?

And why is P multiplied to the left of $[x]_B$?

4.7

BB, SD, EV

$\underset{C \leftarrow B}{P}$ IS AN NXN MATRIX THAT TRANSFORMS $[\vec{x}]_B$ to $[\vec{x}]_C$.

$\underset{C \leftarrow B}{P}$ CAN BE CALCULATED BY TRANSFORMING $\left[\vec{c}_1 \ \vec{c}_2 \,\vdots\, \vec{b}_1 \ \vec{b}_2\right]$ (FOR EXAMPLE) INTO $\left[I \,\vdots\, \underset{C \leftarrow B}{P}\right]$.

EXERCISES 7-10