From Taylor Example 6.2, we arrive at

$$x = \int \sqrt{\frac{y}{2a - y}}\, dy. \qquad (6.23)$$

This integral can be evaluated by the unlikely looking substitution

$$y = a(1 - \cos\theta) \qquad (6.24)$$

which gives (as you should check)

$$x = a \int (1 - \cos\theta)\, d\theta$$

Let's check, as requested:

$y = a(1 - \cos\theta) = a - a\cos\theta$ means $dy = a \sin\theta\, d\theta$

So

$$x = \int \sqrt{\frac{y}{2a - y}}\, dy$$
$$= \int \sqrt{\frac{a(1-\cos\theta)}{2a - a(1-\cos\theta)}}\, a\sin\theta\, d\theta$$
$$= \int \sqrt{\frac{a(1-\cos\theta)}{2a - a + a\cos\theta}}\, a\sin\theta\, d\theta$$
$$= \int \sqrt{\frac{a(1-\cos\theta)}{a + a\cos\theta}}\, a\sin\theta\, d\theta$$
$$= \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\, a\sin\theta\, d\theta$$
$$= \int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\, a\sin\theta\, d\theta$$
$$= a\int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\, \sin\theta\, d\theta$$

This doesn't look particularly simpler, but the symmetry of the expression under the radical is certainly appealing! We can keep going with some algebraic manipulations, and also some trig identities:

$$x = a\int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\, \sin\theta\, d\theta$$
$$= a\int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\, \sqrt{\sin^2\theta}\, d\theta$$
$$= a\int \sqrt{\frac{(1-\cos\theta)\sin^2\theta}{(1+\cos\theta)}}\, d\theta$$
$$= a\int \sqrt{\frac{(1-\cos\theta)\,(1-\cos^2\theta)}{(1+\cos\theta)}}\, d\theta$$
$$= a\int \sqrt{\frac{(1-\cos\theta)\,(1-\cos\theta)\,(1+\cos\theta)}{(1+\cos\theta)}}\, d\theta$$
$$= a\int \sqrt{(1-\cos\theta)\,(1-\cos\theta)}\, d\theta$$
$$\boxed{= a\int (1-\cos\theta)\, d\theta}$$

As requested!