

Chapter Thirteen

A FUNDAMENTAL TOOL: VECTORS

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13.1 DISPLACEMENT VECTORS

Suppose you are a pilot planning a flight from Dallas to Pittsburgh. There are two things you must know: the distance to be traveled (so you have enough fuel to make it) and in what direction to go (so you don't miss Pittsburgh). Both these quantities together specify the displacement or *displacement vector* between the two cities.

The **displacement vector** from one point to another is an arrow with its tail at the first point and its tip at the second. The **magnitude** (or length) of the displacement vector is the distance between the points and is represented by the length of the arrow. The **direction** of the displacement vector is the direction of the arrow.

Figure 13.1 shows a map with the displacement vectors from Dallas to Pittsburgh, from Albuquerque to Oshkosh, and from Los Angeles to Buffalo, SD. These displacement vectors have the same length and the same direction. We say that the displacement vectors between the corresponding cities are the same, even though they do not coincide. In other words

Displacement vectors which point in the same direction and have the same magnitude are considered to be the same, even if they do not coincide.

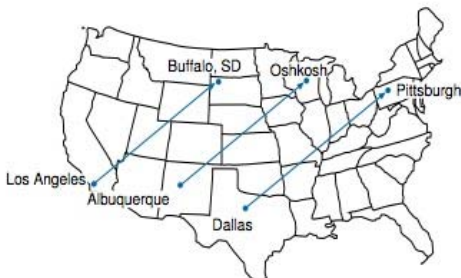


Figure 13.1: Displacement vectors between cities

Notation and Terminology

The displacement vector is our first example of a vector. Vectors have both magnitude and direction; in comparison, a quantity specified only by a number, but no direction, is called a *scalar*.¹ For instance, the time taken by the flight from Dallas to Pittsburgh is a scalar quantity. Displacement is a vector since it requires both distance and direction to specify it.

In this book, vectors are written with an arrow over them, \vec{v} , to distinguish them from scalars. Other books use a bold \mathbf{v} to denote a vector. We use the notation \overrightarrow{PQ} to denote the displacement vector from a point P to a point Q . The magnitude, or length, of a vector \vec{v} is written $\|\vec{v}\|$.

Addition and Subtraction of Displacement Vectors

Suppose NASA commands a robot on Mars to move 75 meters in one direction and then 50 meters in another direction. (See Figure 13.2.) Where does the robot end up? Suppose the displacements are represented by the vectors \vec{v} and \vec{w} , respectively. Then the sum $\vec{v} + \vec{w}$ gives the final position.

¹ So named by W. R. Hamilton because they are merely numbers on the *scale* from $-\infty$ to ∞ .

The **sum**, $\vec{v} + \vec{w}$, of two vectors \vec{v} and \vec{w} is the combined displacement resulting from first applying \vec{v} and then \vec{w} . (See Figure 13.3.) The sum $\vec{w} + \vec{v}$ gives the same displacement.

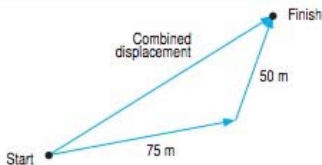


Figure 13.2: Sum of displacements of robots on Mars

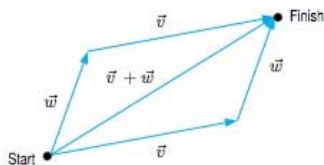


Figure 13.3: The sum $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Suppose two different robots start from the same location. One moves along a displacement vector \vec{v} and the second along a displacement vector \vec{w} . What is the displacement vector, \vec{x} , from the first robot to the second? (See Figure 13.4.) Since $\vec{v} + \vec{x} = \vec{w}$, we define \vec{x} to be the difference $\vec{x} = \vec{w} - \vec{v}$. In other words, $\vec{w} - \vec{v}$ gets you from the first robot to the second.

The **difference**, $\vec{w} - \vec{v}$, is the displacement vector that, when added to \vec{v} , gives \vec{w} . That is, $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$. (See Figure 13.4.)

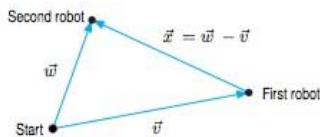


Figure 13.4: The difference $\vec{w} - \vec{v}$

If the robot ends up where it started, then its total displacement vector is the **zero vector**, $\vec{0}$. The zero vector has no direction.

The **zero vector**, $\vec{0}$, is a displacement vector with zero length.

Scalar Multiplication of Displacement Vectors

If \vec{v} represents a displacement vector, the vector $2\vec{v}$ represents a displacement of twice the magnitude in the same direction as \vec{v} . Similarly, $-2\vec{v}$ represents a displacement of twice the magnitude in the opposite direction. (See Figure 13.5.)

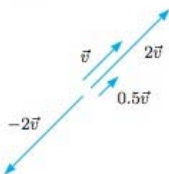


Figure 13.5: Scalar multiples of the vector \vec{v}

If λ is a scalar and \vec{v} is a displacement vector, the **scalar multiple of \vec{v} by λ** , written $\lambda\vec{v}$, is the displacement vector with the following properties:

- The displacement vector $\lambda\vec{v}$ is parallel to \vec{v} , pointing in the same direction if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.
- The magnitude of $\lambda\vec{v}$ is $|\lambda|$ times the magnitude of \vec{v} , that is, $\|\lambda\vec{v}\| = |\lambda| \|\vec{v}\|$.

Note that $|\lambda|$ represents the absolute value of the scalar λ while $\|\lambda\vec{v}\|$ represents the magnitude of the vector $\lambda\vec{v}$.

Example 1 Explain why $\vec{w} - \vec{v} = \vec{w} + (-1)\vec{v}$.

Solution The vector $(-1)\vec{v}$ has the same magnitude as \vec{v} , but points in the opposite direction. Figure 13.6 shows that the combined displacement $\vec{w} + (-1)\vec{v}$ is the same as the displacement $\vec{w} - \vec{v}$.

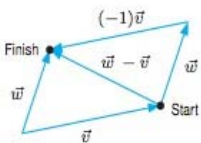


Figure 13.6: Explanation for why $\vec{w} - \vec{v} = \vec{w} + (-1)\vec{v}$

Parallel Vectors

Two vectors \vec{v} and \vec{w} are *parallel* if one is a scalar multiple of the other, that is, if $\vec{w} = \lambda\vec{v}$, for some scalar λ .

Components of Displacement Vectors: The Vectors \vec{i} , \vec{j} , and \vec{k}

Suppose that you live in a city with equally spaced streets running east-west and north-south and that you want to tell someone how to get from one place to another. You'd be likely to tell them how many blocks east-west and how many blocks north-south to go. For example, to get from P to Q in Figure 13.7, we go 4 blocks east and 1 block south. If \vec{i} and \vec{j} are as shown in Figure 13.7, then the displacement vector from P to Q is $4\vec{i} - \vec{j}$.

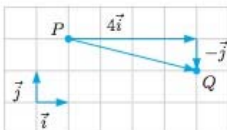


Figure 13.7: The displacement vector from P to Q is $4\vec{i} - \vec{j}$

We extend the same idea to 3 dimensions. First we choose a Cartesian system of coordinate axes. The three vectors of length 1 shown in Figure 13.8 are the vector \vec{i} , which points along the positive x -axis, the vector \vec{j} , along the positive y -axis, and the vector \vec{k} , along the positive z -axis.

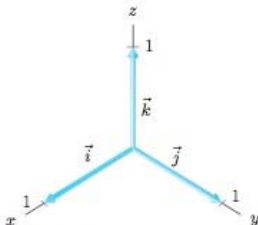


Figure 13.8: The vectors \vec{i} , \vec{j} and \vec{k} in 3-space

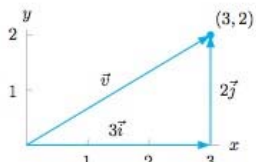


Figure 13.9: We resolve \vec{v} into components by writing $\vec{v} = 3\vec{i} + 2\vec{j}$

Writing Displacement Vectors Using \vec{i} , \vec{j} , \vec{k}

Any displacement in 3-space or the plane can be expressed as a combination of displacements in the coordinate directions. For example, Figure 13.9 shows that the displacement vector \vec{v} from the origin to the point $(3, 2)$ can be written as a sum of displacement vectors along the x - and y -axes:

$$\vec{v} = 3\vec{i} + 2\vec{j}.$$

This is called *resolving \vec{v} into components*. In general:

We **resolve** \vec{v} into components by writing \vec{v} in the form

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k},$$

where v_1, v_2, v_3 are scalars. We call $v_1\vec{i}$, $v_2\vec{j}$, and $v_3\vec{k}$ the **components** of \vec{v} .

An Alternative Notation for Vectors

Many people write a vector in three dimensions as a string of three numbers, that is, as

$$\vec{v} = (v_1, v_2, v_3) \quad \text{instead of} \quad \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.$$

Since the first notation can be confused with a point and the second cannot, we usually use the second form.

Example 2 Resolve the displacement vector, \vec{v} , from the point $P_1 = (2, 4, 10)$ to the point $P_2 = (3, 7, 6)$ into components.

Solution To get from P_1 to P_2 , we move 1 unit in the positive x -direction, 3 units in the positive y -direction, and 4 units in the negative z -direction. Hence $\vec{v} = \vec{i} + 3\vec{j} - 4\vec{k}$.

Example 3 Decide whether the vector $\vec{v} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ is parallel to each of the following vectors:

$$\vec{w} = 4\vec{i} + 6\vec{j} + 10\vec{k}, \quad \vec{a} = -\vec{i} - 1.5\vec{j} - 2.5\vec{k}, \quad \vec{b} = 4\vec{i} + 6\vec{j} + 9\vec{k}.$$

Solution Since $\vec{w} = 2\vec{v}$ and $\vec{a} = -0.5\vec{v}$, the vectors \vec{v} , \vec{w} , and \vec{a} are parallel. However, \vec{b} is not a multiple of \vec{v} (since, for example, $4/2 \neq 9/5$), so \vec{v} and \vec{b} are not parallel.

In general, Figure 13.10 shows us how to express the displacement vector between two points in components:

Components of Displacement Vectors

The displacement vector from the point $P_1 = (x_1, y_1, z_1)$ to the point $P_2 = (x_2, y_2, z_2)$ is given in components by

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}.$$

Position Vectors: Displacement of a Point from the Origin

A displacement vector whose tail is at the origin is called a *position vector*. Thus, any point (x_0, y_0, z_0) in space has associated with it the position vector $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$. (See Figure 13.11.) In general, a position vector gives the displacement of a point from the origin.

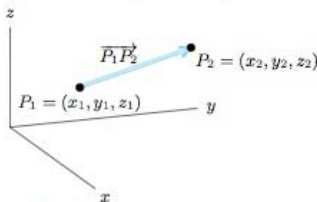


Figure 13.10: The displacement vector $\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

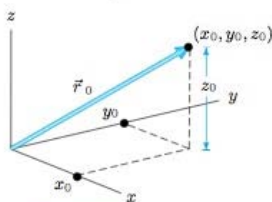


Figure 13.11: The position vector $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$

The Components of the Zero Vector

The zero displacement vector has magnitude equal to zero and is written $\vec{0}$. So $\vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}$.

The Magnitude of a Vector in Components

For a vector, $\vec{v} = v_1\vec{i} + v_2\vec{j}$, the Pythagorean theorem is used to find its magnitude, $\|\vec{v}\|$. (See Figure 13.12.) The angle θ gives the direction of \vec{v} .

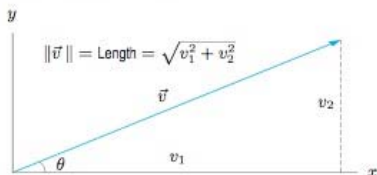


Figure 13.12: Magnitude, $\|\vec{v}\|$, of a 2-dimensional vector, \vec{v}

In three dimensions, for a vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, we have

$$\text{Magnitude of } \vec{v} = \|\vec{v}\| = \text{Length of the arrow} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

For instance, if $\vec{v} = 3\vec{i} - 4\vec{j} + 5\vec{k}$, then $\|\vec{v}\| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{50}$.

Addition and Scalar Multiplication of Vectors in Components

Suppose the vectors \vec{v} and \vec{w} are given in components:

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \quad \text{and} \quad \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}.$$

Then

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{i} + (v_2 + w_2)\vec{j} + (v_3 + w_3)\vec{k},$$

and

$$\lambda\vec{v} = \lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}.$$

Figures 13.13 and 13.14 illustrate these properties in two dimensions. Finally, $\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w}$, so we can write $\vec{v} - \vec{w} = (v_1 - w_1)\vec{i} + (v_2 - w_2)\vec{j} + (v_3 - w_3)\vec{k}$.

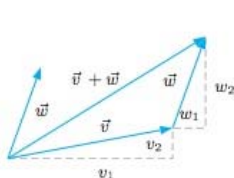


Figure 13.13: Sum $\vec{v} + \vec{w}$ in components

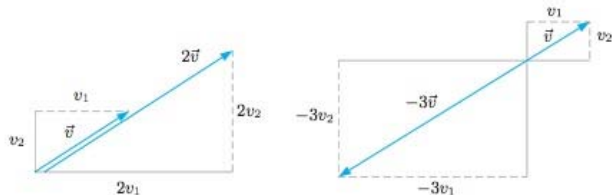


Figure 13.14: Scalar multiples of vectors showing \vec{v} , $2\vec{v}$, and $-3\vec{v}$

How to Resolve a Vector into Components

You may wonder how we find the components of a 2-dimensional vector, given its length and direction. Suppose the vector \vec{v} has length v and makes an angle of θ with the x -axis, measured counterclockwise, as in Figure 13.15. If $\vec{v} = v_1\vec{i} + v_2\vec{j}$, Figure 13.15 shows that

$$v_1 = v \cos \theta \quad \text{and} \quad v_2 = v \sin \theta.$$

Thus, we resolve \vec{v} into components by writing

$$\vec{v} = (v \cos \theta)\vec{i} + (v \sin \theta)\vec{j}.$$

Vectors in 3-space are resolved using direction cosines; see Problem 58 on page 754.

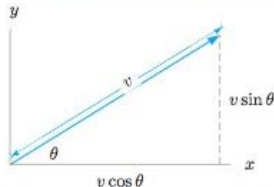


Figure 13.15: Resolving a vector: $\vec{v} = (v \cos \theta)\vec{i} + (v \sin \theta)\vec{j}$

Example 4 Resolve \vec{v} into components if $\|\vec{v}\| = 2$ and $\theta = \pi/6$.

Solution We have $\vec{v} = 2 \cos(\pi/6)\vec{i} + 2 \sin(\pi/6)\vec{j} = 2(\sqrt{3}/2)\vec{i} + 2(1/2)\vec{j} = \sqrt{3}\vec{i} + \vec{j}$.

Unit Vectors

A *unit vector* is a vector whose magnitude is 1. The vectors \vec{i} , \vec{j} , and \vec{k} are unit vectors in the directions of the coordinate axes. It is often helpful to find a unit vector in the same direction as a given vector \vec{v} . Suppose that $\|\vec{v}\| = 10$; a unit vector in the same direction as \vec{v} is $\vec{v}/10$. In general, a unit vector in the direction of any nonzero vector \vec{v} is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Example 5 Find a unit vector, \vec{u} , in the direction of the vector $\vec{v} = \vec{i} + 3\vec{j}$.

Solution If $\vec{v} = \vec{i} + 3\vec{j}$, then $\|\vec{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Thus, a unit vector in the same direction is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{10}}(\vec{i} + 3\vec{j}) = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j} \approx 0.32\vec{i} + 0.95\vec{j}.$$

Example 6 Find a unit vector at the point (x, y, z) that points radially outward away from the origin.

Solution The vector from the origin to (x, y, z) is the position vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Thus, if we put its tail at (x, y, z) it will point away from the origin. Its magnitude is

$$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2},$$

so a unit vector pointing in the same direction is

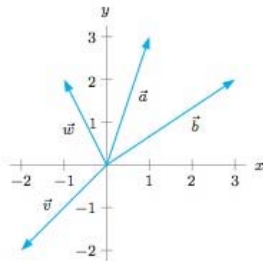
$$\frac{\vec{r}}{\|\vec{r}\|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\vec{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\vec{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\vec{k}.$$

Exercises and Problems for Section 13.1

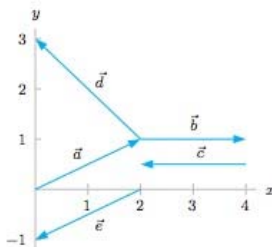
Exercises

In Exercises 1–6, resolve the vectors into components.

1.



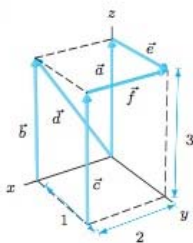
2.



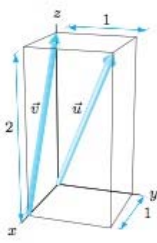
3. A vector starting at the point $Q = (4, 6)$ and ending at the point $P = (1, 2)$.

4. A vector starting at the point $P = (1, 2)$ and ending at the point $Q = (4, 6)$.

5.



6.



17. $\vec{v} = \vec{i} - \vec{j} + 3\vec{k}$

18. $\vec{v} = 7.2\vec{i} - 1.5\vec{j} + 2.1\vec{k}$

19. $\vec{v} = 1.2\vec{i} - 3.6\vec{j} + 4.1\vec{k}$

For Exercises 20–25, perform the indicated operations on the following vectors:

$$\vec{a} = 2\vec{j} + \vec{k}, \quad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k}, \quad \vec{c} = \vec{i} + 6\vec{j},$$

$$\vec{x} = -2\vec{i} + 9\vec{j}, \quad \vec{y} = 4\vec{i} - 7\vec{j}, \quad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}.$$

For Exercises 7–14, perform the indicated computation.

7. $(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j})$

8. $(\vec{i} + 2\vec{j}) + (-3)(2\vec{i} + \vec{j})$

9. $-4(\vec{i} - 2\vec{j}) - 0.5(\vec{i} - \vec{k})$

10. $2(0.45\vec{i} - 0.9\vec{j} - 0.01\vec{k}) - 0.5(1.2\vec{i} - 0.1\vec{k})$

11. $(3\vec{i} - 4\vec{j} + 2\vec{k}) - (6\vec{i} + 8\vec{j} - \vec{k})$

12. $(4\vec{i} - 3\vec{j} + 7\vec{k}) - 2(5\vec{i} + \vec{j} - 2\vec{k})$

13. $(0.6\vec{i} + 0.2\vec{j} - \vec{k}) + (0.3\vec{i} + 0.3\vec{k})$

14. $\frac{1}{2}(2\vec{i} - \vec{j} + 3\vec{k}) + 3(\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{2}\vec{k})$

In Exercises 15–19, find the length of the vectors.

15. $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$

16. $\vec{z} = \vec{i} - 3\vec{j} - \vec{k}$

20. $4\vec{z}$

21. $5\vec{a} + 2\vec{b}$

22. $\vec{a} + \vec{z}$

23. $2\vec{c} + \vec{x}$

24. $2\vec{a} + 7\vec{b} - 5\vec{z}$

25. $\|\vec{y} - \vec{x}\|$

26. (a) Draw the position vector for $\vec{v} = 5\vec{i} - 7\vec{j}$.

(b) What is $\|\vec{v}\|$?(c) Find the angle between \vec{v} and the positive x -axis.

27. Find the unit vector in the direction of $0.06\vec{i} - 0.08\vec{k}$.

28. Find the unit vector in the opposite direction to $\vec{i} - \vec{j} + \vec{k}$.

29. Find a unit vector in the opposite direction to $2\vec{i} - \vec{j} - \sqrt{11}\vec{k}$.

30. Find a vector with length 2 that points in the same direction as $\vec{i} - \vec{j} + 2\vec{k}$.

Problems

31. Find the value(s) of a making $\vec{v} = 5a\vec{i} - 3\vec{j}$ parallel to $\vec{w} = a^2\vec{i} + 6\vec{j}$.

32. (a) Find a unit vector from the point $P = (1, 2)$ and toward the point $Q = (4, 6)$.

(b) Find a vector of length 10 pointing in the same direction.

33. If north is the direction of the positive y -axis and east is the direction of the positive x -axis, give the unit vector pointing northwest.

34. Resolve the following vectors into components:

(a) The vector in 2-space of length 2 pointing up and to the right at an angle of $\pi/4$ with the x -axis.

(b) The vector in 3-space of length 1 lying in the xz -plane pointing upward at an angle of $\pi/6$ with the positive x -axis.

35. (a) From Figure 13.16, read off the coordinates of the five points, A, B, C, D, E , and thus resolve into components the following two vectors: $\vec{u} = (2.5)\vec{AB} + (-0.8)\vec{CD}$, $\vec{v} = (2.5)\vec{BA} - (-0.8)\vec{CD}$

(b) What is the relation between \vec{u} and \vec{v} ? Why was this to be expected?

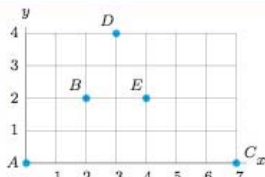


Figure 13.16

36. Find the components of a vector \vec{p} that has the same direction as \vec{EA} in Figure 13.16 and whose length equals two units.

37. For each of the four statements below, answer the following questions: Does the statement make sense? If yes, is it true for all possible choices of \vec{a} and \vec{b} ? If no, why not?

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(b) $\vec{a} + \|\vec{b}\| = \|\vec{a} + \vec{b}\|$

(c) $\|\vec{b} + \vec{a}\| = \|\vec{a} + \vec{b}\|$

(d) $\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$.

38. Two adjacent sides of a regular hexagon are given as the vectors \vec{u} and \vec{v} in Figure 13.17. Label the remaining sides in terms of \vec{u} and \vec{v} .

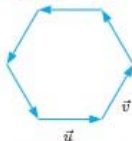


Figure 13.17

39. For what values of t are the following pairs of vectors parallel?
- (a) $2\vec{i} + (t^2 + \frac{2}{3}t + 1)\vec{j} + t\vec{k}$, $6\vec{i} + 8\vec{j} + 3\vec{k}$
 (b) $t\vec{i} + \vec{j} + (t-1)\vec{k}$, $2\vec{i} - 4\vec{j} + \vec{k}$
 (c) $2t\vec{i} + t\vec{j} + t\vec{k}$, $6\vec{i} + 3\vec{j} + 3\vec{k}$.
40. Find all vectors \vec{v} in 2 dimensions having $\|\vec{v}\| = 5$ such that the \vec{i} -component of \vec{v} is $3\vec{i}$.

41. Find all vectors \vec{v} in the plane such that $\|\vec{v}\| = 1$ and $\|\vec{v} + \vec{i}\| = 1$.
42. Figure 13.18 shows a molecule with four atoms at O , A , B and C . Check that every atom in the molecule is 2 units away from every other atom.

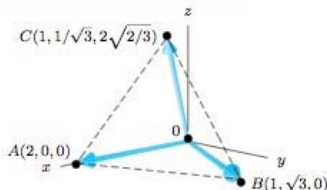


Figure 13.18

43. Show that the medians of a triangle intersect at a point $\frac{1}{3}$ of the way along each median from the side it bisects.

Strengthen Your Understanding

In Problems 44–47, explain what is wrong with the statement.

44. If $\|\vec{u}\| = 1$ and $\|\vec{v}\| > 0$, then $\|\vec{u} + \vec{v}\| \geq 1$.
 45. The vector $c\vec{u}$ has the same direction as \vec{u} .
 46. $\|\vec{v} - \vec{u}\|$ is the length of the shorter of the two diagonals of the parallelogram determined by \vec{u} and \vec{v} .
 47. Given three vectors \vec{u} , \vec{v} , and \vec{w} , if $\vec{u} + \vec{w} = \vec{u}$ then it is possible for $\vec{v} + \vec{w} \neq \vec{v}$.

In Problems 48–50, give an example of:

48. A vector \vec{v} of length 2 with a positive \vec{k} -component and lying on a plane parallel to the yz -plane.
 49. Two unit vectors \vec{u} and \vec{v} for which $\vec{v} - \vec{u}$ is also a unit vector.
 50. Two vectors \vec{u} and \vec{v} that have difference vector $\vec{w} = 2\vec{i} + 3\vec{j}$.

Are the statements in Problems 51–60 true or false? Give reasons for your answer.

51. There is exactly one unit vector parallel to a given nonzero vector \vec{v} .
 52. The vector $\frac{1}{\sqrt{3}}\vec{i} + \frac{-1}{\sqrt{3}}\vec{j} + \frac{2}{\sqrt{3}}\vec{k}$ is a unit vector.
 53. The length of the vector $2\vec{v}$ is twice the length of the vector \vec{v} .
 54. If \vec{v} and \vec{w} are any two vectors, then $\|\vec{v} + \vec{w}\| = \|\vec{v}\| + \|\vec{w}\|$.
 55. If \vec{v} and \vec{w} are any two vectors, then $\|\vec{v} - \vec{w}\| = \|\vec{v}\| - \|\vec{w}\|$.
 56. The vectors $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$ are parallel.
 57. The vector $\vec{u} + \vec{v}$ is always larger in magnitude than both \vec{u} and \vec{v} .
 58. For any scalar c and vector \vec{v} we have $\|c\vec{v}\| = c\|\vec{v}\|$.
 59. The displacement vector from $(1, 1, 1)$ to $(1, 2, 3)$ is $-\vec{j} - 2\vec{k}$.
 60. The displacement vector from (a, b) to (c, d) is the same as the displacement vector from (c, d) to (a, b) .

13.2 VECTORS IN GENERAL

Besides displacement, there are many quantities that have both magnitude and direction and are added and multiplied by scalars in the same way as displacements. Any such quantity is called a *vector* and is represented by an arrow in the same manner we represent displacements. The length of the arrow is the *magnitude* of the vector, and the direction of the arrow is the direction of the vector.

Velocity Versus Speed

The speed of a moving body tells us how fast it is moving, say 80 km/hr. The speed is just a number; it is therefore a scalar. The velocity, on the other hand, tells us both how fast the body is moving and the direction of motion; it is a vector. For instance, if a car is heading northeast at 80 km/hr, then its velocity is a vector of length 80 pointing northeast.

The **velocity vector** of a moving object is a vector whose magnitude is the speed of the object and whose direction is the direction of its motion.

The velocity vector is the displacement vector if the object moves at constant velocity for one unit of time.

Example 1 A car is traveling north at a speed of 100 km/hr, while a plane above is flying horizontally southwest at a speed of 500 km/hr. Draw the velocity vectors of the car and the plane.

Solution Figure 13.19 shows the velocity vectors. The plane's velocity vector is five times as long as the car's, because its speed is five times as great.

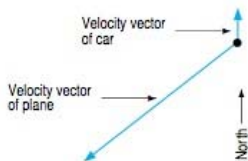


Figure 13.19: Velocity vector of the car is 100 km/hr north and of the plane is 500 km/hr southwest

The next example illustrates that the velocity vectors for two motions add to give the velocity vector for the combined motion, just as displacements do.

Example 2 A riverboat is moving with velocity \vec{v} and a speed of 8 km/hr relative to the water. In addition, the river has a current \vec{c} and a speed of 1 km/hr. (See Figure 13.20.) What is the physical significance of the vector $\vec{v} + \vec{c}$?

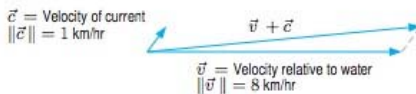


Figure 13.20: Boat's velocity relative to the river bed is the sum, $\vec{v} + \vec{c}$

Solution The vector \vec{v} shows how the boat is moving relative to the water, while \vec{c} shows how the water is moving relative to the riverbed. During an hour, imagine that the boat first moves 8 km relative to the water, which remains still; this displacement is represented by \vec{v} . Then imagine the water moving 1 km while the boat remains stationary relative to the water; this displacement is represented by \vec{c} . The combined displacement is represented by $\vec{v} + \vec{c}$. Thus, the vector $\vec{v} + \vec{c}$ is the velocity of the boat relative to the riverbed.

Note that the effective speed of the boat is not necessarily 9 km/hr unless the boat is moving in the direction of the current. Although we add the velocity vectors, we do not necessarily add their lengths.

Scalar multiplication also makes sense for velocity vectors. For example, if \vec{v} is a velocity vector, then $-2\vec{v}$ represents a velocity of twice the magnitude in the opposite direction.

Example 3 A ball is moving with velocity \vec{v} when it hits a wall at a right angle and bounces straight back, with its speed reduced by 20%. Express its new velocity in terms of the old one.

Solution The new velocity is $-0.8\vec{v}$, where the negative sign expresses the fact that the new velocity is in the direction opposite to the old.

We can represent velocity vectors in components in the same way we did on page 723.

Example 4 Represent the velocity vectors of the car and the plane in Example 1 using components. Take north to be the positive y -axis, east to be the positive x -axis, and upward to be the positive z -axis.

Solution The car is traveling north at 100 km/hr, so the y -component of its velocity is $100\vec{j}$ and the x -component is $0\vec{i}$. Since it is traveling horizontally, the z -component is $0\vec{k}$. So we have

$$\text{Velocity of car} = 0\vec{i} + 100\vec{j} + 0\vec{k} = 100\vec{j}.$$

The plane's velocity vector also has \vec{k} component equal to zero. Since it is traveling southwest, its \vec{i} and \vec{j} components have negative coefficients (north and east are positive). Since the plane is traveling at 500 km/hr, in one hour it is displaced $500/\sqrt{2} \approx 354$ km to the west and 354 km to the south. (See Figure 13.21.) Thus,

$$\text{Velocity of plane} = -(500 \cos 45^\circ)\vec{i} - (500 \sin 45^\circ)\vec{j} \approx -354\vec{i} - 354\vec{j}.$$

Of course, if the car were climbing a hill or if the plane were descending for a landing, then the \vec{k} component would not be zero.

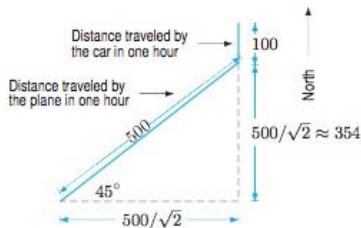


Figure 13.21: Distance traveled by the plane and car in one hour

Acceleration

Another example of a vector quantity is acceleration. Acceleration, like velocity, is specified by both a magnitude and a direction — for example, the acceleration due to gravity is 9.81 m/sec^2 vertically downward.

Force

Force is another example of a vector quantity. Suppose you push on an open door. The result depends both on how hard you push and in what direction. Thus, to specify a force we must give its magnitude (or strength) and the direction in which it is acting. For example, the gravitational force exerted on an object by the earth is a vector pointing from the object toward the center of the earth; its magnitude is the strength of the gravitational force.

Example 5 The earth travels around the sun in an ellipse. The gravitational force on the earth and the velocity of the earth are governed by the following laws:

Newton's Law of Gravitation: The gravitational attraction, \vec{F} , of a mass m_1 on a mass m_2 at a

distance r has magnitude $\|\vec{F}\| = Gm_1m_2/r^2$, where G is a constant, and is directed from m_2 toward m_1 .

Kepler's Second Law: The line joining a planet to the sun sweeps out equal areas in equal times.

- (a) Sketch vectors representing the gravitational force of the sun on the earth at two different positions in the earth's orbit.
 (b) Sketch the velocity vector of the earth at two points in its orbit.

Solution

- (a) Figure 13.22 shows the earth orbiting the sun. Note that the gravitational force vector always points toward the sun and is larger when the earth is closer to the sun because of the r^{-2} term in the denominator. (In fact, the real orbit looks much more like a circle than we have shown here.)
 (b) The velocity vector points in the direction of motion of the earth. Thus, the velocity vector is tangent to the ellipse. See Figure 13.23. Furthermore, the velocity vector is longer at points of the orbit where the planet is moving quickly, because the magnitude of the velocity vector is the speed. Kepler's Second Law enables us to determine when the earth is moving quickly and when it is moving slowly. Over a fixed period of time, say one month, the line joining the earth to the sun sweeps out a sector having a certain area. Figure 13.23 shows two sectors swept out in two different one-month time-intervals. Kepler's law says that the areas of the two sectors are the same. Thus, the earth must move farther in a month when it is close to the sun than when it is far from the sun. Therefore, the earth moves faster when it is closer to the sun and slower when it is farther away.

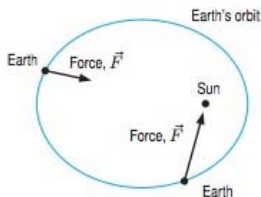


Figure 13.22: Gravitational force, \vec{F} , exerted by the sun on the earth: Greater magnitude closer to sun

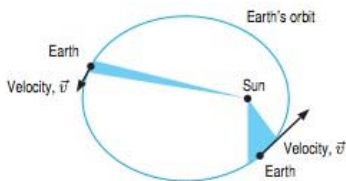


Figure 13.23: The velocity vector, \vec{v} , of the earth: Greater magnitude closer to the sun

Properties of Addition and Scalar Multiplication

In general, vectors add, subtract, and are multiplied by scalars in the same way as displacement vectors. Thus, for any vectors \vec{u} , \vec{v} , and \vec{w} and any scalars α and β , we have the following properties:

Commutativity

$$1. \vec{v} + \vec{w} = \vec{w} + \vec{v}$$

Distributivity

$$4. (\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$$

$$5. \alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$$

Associativity

$$2. (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$3. \alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$$

Identity

$$6. 1\vec{v} = \vec{v}$$

$$8. \vec{v} + \vec{0} = \vec{v}$$

$$7. 0\vec{v} = \vec{0}$$

$$9. \vec{w} + (-1)\vec{v} = \vec{w} - \vec{v}$$

Problems 30–37 at the end of this section ask for a justification of these results in terms of displacement vectors.

Using Components

Example 6 A plane, heading due east at an airspeed of 600 km/hr, experiences a wind of 50 km/hr blowing toward the northeast. Find the plane's direction and ground speed.

Solution We choose a coordinate system with the x -axis pointing east and the y -axis pointing north. See Figure 13.24.

The airspeed tells us the speed of the plane relative to still air. Thus, the plane is moving due east with velocity $\vec{v} = 600\vec{i}$ relative to still air. In addition, the air is moving with a velocity \vec{w} . Writing \vec{w} in components, we have

$$\vec{w} = (50\cos 45^\circ)\vec{i} + (50\sin 45^\circ)\vec{j} = 35.4\vec{i} + 35.4\vec{j}.$$

The vector $\vec{v} + \vec{w}$ represents the displacement of the plane in one hour relative to the ground. Therefore, $\vec{v} + \vec{w}$ is the velocity of the plane relative to the ground. In components, we have

$$\vec{v} + \vec{w} = 600\vec{i} + (35.4\vec{i} + 35.4\vec{j}) = 635.4\vec{i} + 35.4\vec{j}.$$

The direction of the plane's motion relative to the ground is given by the angle θ in Figure 13.24, where

$$\tan \theta = \frac{35.4}{635.4}$$

so

$$\theta = \arctan\left(\frac{35.4}{635.4}\right) = 3.2^\circ.$$

The ground speed is the speed of the plane relative to the ground, so

$$\text{Groundspeed} = \sqrt{635.4^2 + 35.4^2} = 636.4 \text{ km/hr.}$$

Thus, the speed of the plane relative to the ground has been increased slightly by the wind. (This is as we would expect, as the wind has a positive component in the direction in which the plane is traveling.) The angle θ shows how far the plane is blown off course by the wind.

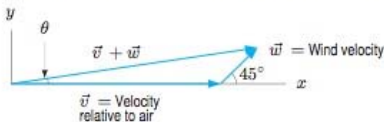


Figure 13.24: Plane's velocity relative to the ground is the sum $\vec{v} + \vec{w}$

Vectors in n Dimensions

Using the alternative notation $\vec{v} = (v_1, v_2, v_3)$ for a vector in 3-space, we can define a vector in n dimensions as a string of n numbers. Thus, a vector in n dimensions can be written as

$$\vec{c} = (c_1, c_2, \dots, c_n).$$

Addition and scalar multiplication are defined by the formulas

$$\vec{v} + \vec{w} = (v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$$

and

$$\lambda \vec{v} = \lambda(v_1, v_2, \dots, v_n) = (\lambda v_1, \lambda v_2, \dots, \lambda v_n).$$

Why Do We Want Vectors in n Dimensions?

Vectors in two and three dimensions can be used to model displacement, velocities, or forces. But what about vectors in n dimensions? There is another interpretation of 3-dimensional vectors (or 3-vectors) that is useful: they can be thought of as listing three different quantities — for example, the displacements parallel to the x -, y -, and z -axes. Similarly, the n -vector

$$\vec{c} = (c_1, c_2, \dots, c_n)$$

can be thought of as a way of keeping n different quantities organized. For example, a *population* vector \vec{N} shows the number of children and adults in a population:

$$\vec{N} = (\text{Number of children, Number of adults}),$$

or, if we are interested in a more detailed breakdown of ages, we might give the number in each ten-year age bracket in the population (up to age 110) in the form

$$\vec{N} = (N_1, N_2, N_3, N_4, \dots, N_{10}, N_{11}),$$

where N_1 is the population aged 0–9, and N_2 is the population aged 10–19, and so on.

A *consumption* vector,

$$\vec{q} = (q_1, q_2, \dots, q_n)$$

shows the quantities q_1, q_2, \dots, q_n consumed of each of n different goods. A *price* vector

$$\vec{p} = (p_1, p_2, \dots, p_n)$$

contains the prices of n different items.

In 1907, Hermann Minkowski used vectors with four components when he introduced *space-time coordinates*, whereby each event is assigned a vector position \vec{v} with four coordinates, three for its position in space and one for time:

$$\vec{v} = (x, y, z, t).$$

Example 7 Suppose the vector \vec{I} represents the number of copies, in thousands, made by each of four copy centers in the month of December and \vec{J} represents the number of copies made at the same four copy centers during the previous eleven months (the “year-to-date”). If $\vec{I} = (25, 211, 818, 642)$, and $\vec{J} = (331, 3227, 1377, 2570)$, compute $\vec{I} + \vec{J}$. What does this sum represent?

Solution The sum is

$$\vec{I} + \vec{J} = (25 + 331, 211 + 3227, 818 + 1377, 642 + 2570) = (356, 3438, 2195, 3212).$$

Each term in $\vec{I} + \vec{J}$ represents the sum of the number of copies made in December plus those in the previous eleven months, that is, the total number of copies made during the entire year at that particular copy center.

Example 8 The price vector $\vec{p} = (p_1, p_2, p_3)$ represents the prices in dollars of three goods. Write a vector that gives the prices of the same goods in cents.

Solution The prices in cents are $100p_1$, $100p_2$, and $100p_3$ respectively, so the new price vector is

$$(100p_1, 100p_2, 100p_3) = 100\vec{p}.$$

Exercises and Problems for Section 13.2

Exercises

In Exercises 1–5, say whether the given quantity is a vector or a scalar.

- The population of the US.
- The distance from Seattle to St. Louis.
- The temperature at a point on the earth's surface.
- The magnetic field at a point on the earth's surface.
- The populations of each of the 50 states.
- Give the components of the velocity vector for wind blowing at 10 km/hr toward the southeast. (Assume north is in the positive y -direction.)
- Give the components of the velocity vector of a boat that is moving at 40 km/hr in a direction 20° south of west. (Assume north is in the positive y -direction.)

- A car is traveling at a speed of 50 km/hr. The positive y -axis is north and the positive x -axis is east. Resolve the car's velocity vector (in 2-space) into components if the car is traveling in each of the following directions:

- | | |
|---------------|---------------|
| (a) East | (b) South |
| (c) Southeast | (d) Northwest |

- Which is traveling faster, a car whose velocity vector is $21\vec{i} + 35\vec{j}$, or a car whose velocity vector is $40\vec{i}$, assuming that the units are the same for both directions?
- What angle does a force of $\vec{F} = 15\vec{i} + 18\vec{j}$ make with the x -axis?

Problems

- The velocity of the current in a river is $\vec{c} = 0.6\vec{i} + 0.8\vec{j}$ km/hr. A boat moves relative to the water with velocity $\vec{v} = 8\vec{i}$ km/hr.
 - What is the speed of the boat relative to the riverbed?
 - What angle does the velocity of the boat relative to the riverbed make with the vector \vec{v} ? What does this angle tell us in practical terms?
- Suppose the current in Problem 11 is twice as fast and in the opposite direction. What is the speed of the boat with respect to the riverbed?
- A boat is heading due east at 25 km/hr (relative to the water). The current is moving toward the southwest at 10 km/hr.
 - Give the vector representing the actual movement of the boat.
 - How fast is the boat going, relative to the ground?
 - By what angle does the current push the boat off of its due east course?
- A truck is traveling due north at 30 km/hr approaching a crossroad. On a perpendicular road a police car is traveling west toward the intersection at 40 km/hr. Both vehicles will reach the crossroad in exactly one hour. Find the vector currently representing the displacement of the truck with respect to the police car.
- An airplane heads northeast at an airspeed of 700 km/hr, but there is a wind blowing from the west at 60 km/hr. In what direction does the plane end up flying? What is its speed relative to the ground?
- Two forces, represented by the vectors $\vec{F}_1 = 8\vec{i} - 6\vec{j}$ and $\vec{F}_2 = 3\vec{i} + 2\vec{j}$, are acting on an object. Give a vector representing the force that must be applied to the object if it is to remain stationary.
- An airplane is flying at an airspeed of 500 km/hr in a wind blowing at 60 km/hr toward the southeast. In what

direction should the plane head to end up going due east? What is the airplane's speed relative to the ground?

- An airplane is flying at an airspeed of 600 km/hr in a cross-wind that is blowing from the northeast at a speed of 50 km/hr. In what direction should the plane head to end up going due east?
- The current in a river is pushing a boat in direction 25° north of east with a speed of 12 km/hr. The wind is pushing the same boat in a direction 80° south of east with a speed of 7 km/hr. Find the velocity vector of the boat's engine (relative to the water) if the boat actually moves due east at a speed of 40 km/hr relative to the ground.
- A man wishes to row the shortest possible distance from north to south across a river that is flowing at 4 km/hr from the east. He can row at 5 km/hr.
 - In which direction should he steer?
 - If there is a wind of 10 km/hr from the southwest, in which direction should he steer to try and go directly across the river? What happens?
- A large ship is being towed by two tugs. The larger tug exerts a force which is 25% greater than the smaller tug and at an angle of 30 degrees north of east. Which direction must the smaller tug pull to ensure that the ship travels due east?
- An object P is pulled by a force \vec{F}_1 of magnitude 15 lb at an angle of 20° degrees north of east. In what direction must a force \vec{F}_2 of magnitude 20 lb pull to ensure that P moves due east?
- An object is to be moved vertically upward by a crane. As the crane cannot get directly above the object, three ropes are attached to guide the object. One rope is pulled parallel to the ground with a force of 100 newtons in a direction 30° north of east. The second rope is pulled parallel to the ground with a force of 70 newtons in a direction 80° south of east. If the crane is attached to the

third rope and can pull with a total force of 3000 newtons, find the force vector for the crane. What is the resulting (total) force on the object? (Assume vector \vec{i} points east, vector \vec{j} points north, and vector \vec{k} points vertically up.)

24. The earth is at the origin, the moon is at the point $(384, 0)$, and a spaceship is at $(280, 90)$, where distance is in thousands of kilometers.
- What is the displacement vector of the moon relative to the earth? Of the spaceship relative to the earth? Of the spaceship relative to the moon?
 - How far is the spaceship from the earth? From the moon?
 - The gravitational force on the spaceship from the earth is 461 newtons and from the moon is 26 newtons. What is the resulting force?
25. A particle moving with speed v hits a barrier at an angle of 60° and bounces off at an angle of 60° in the opposite direction with speed reduced by 20 percent. See Figure 13.25. Find the velocity vector of the object after impact.

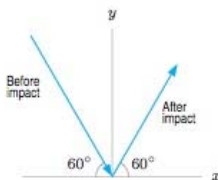


Figure 13.25

26. There are five students in a class. Their scores on the midterm (out of 100) are given by the vector $\vec{v} = (73, 80, 91, 65, 84)$. Their scores on the final (out of 100) are given by $\vec{w} = (82, 79, 88, 70, 92)$. If the final counts twice as much as the midterm, find a vector giving the total scores (as a percentage) of the students.
27. The price vector of beans, rice, and tofu is $(0.30, 0.20, 0.50)$ in dollars per pound. Express it in dollars per ounce.

Strengthen Your Understanding

In Problems 39–40, explain what is wrong with the statement.

39. Two vectors in 3-space that have equal \vec{k} -components and the same magnitude must be the same vector.
40. A vector \vec{v} in the plane whose \vec{i} -component is 0.5 has smaller magnitude than the vector $\vec{w} = 2\vec{i}$.

In Problems 41–42, give an example of:

41. A non-zero vector \vec{F} on the plane that when combined with the force vector $\vec{G} = \vec{i} + \vec{j}$ results in a combined force vector \vec{H} with a positive \vec{i} -component and a negative \vec{j} -component.

28. An object is moving counterclockwise at a constant speed around the circle $x^2 + y^2 = 1$, where x and y are measured in meters. It completes one revolution every minute.

- What is its speed?
- What is its velocity vector 30 seconds after it passes the point $(1, 0)$? Does your answer change if the object is moving clockwise? Explain.

29. An object is attached by a string to a fixed point and rotates 30 times per minute in a horizontal plane. Show that the speed of the object is constant but the velocity is not. What does this imply about the acceleration?

In Problems 30–37, use the geometric definition of addition and scalar multiplication to explain each of the properties.

30. $\vec{w} + \vec{v} = \vec{v} + \vec{w}$ 31. $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

32. $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$ 33. $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

34. $\vec{v} + \vec{0} = \vec{v}$ 35. $1\vec{v} = \vec{v}$

36. $\vec{v} + (-1)\vec{v} = \vec{v} - \vec{v}$

37. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

38. In the game of laser tag, you shoot a harmless laser gun and try to hit a target worn at the waist by other players. Suppose you are standing at the origin of a three-dimensional coordinate system and that the xy -plane is the floor. Suppose that waist-high is 3 feet above floor level and that eye level is 5 feet above the floor. Three of your friends are your opponents. One is standing so that his target is 30 feet along the x -axis, another lying down so that his target is at the point $x = 20, y = 15$, and the third lying in ambush so that his target is at a point 8 feet above the point $x = 12, y = 30$.

- If you aim with your gun at eye level, find the vector from your gun to each of the three targets.
- If you shoot from waist height, with your gun one foot to the right of the center of your body as you face along the x -axis, find the vector from your gun to each of the three targets.

42. Non-zero vectors \vec{u} and \vec{v} such that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.

In Problems 43–48, is the quantity a vector? Give a reason for your answer.

43. Velocity 44. Speed 45. Force

46. Area 47. Acceleration 48. Volume

13.3 THE DOT PRODUCT

We have seen how to add vectors; can we multiply two vectors together? In the next two sections we will see two different ways of doing so: the *scalar product* (or *dot product*), which produces a scalar, and the *vector product* (or *cross product*), which produces a vector.

Definition of the Dot Product

The dot product links geometry and algebra. We already know how to calculate the length of a vector from its components; the dot product gives us a way of computing the angle between two vectors. For any two vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$, shown in Figure 13.26, we define a scalar as follows:

The following two definitions of the **dot product**, or **scalar product**, $\vec{v} \cdot \vec{w}$, are equivalent:

- **Geometric definition**

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta \quad \text{where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{w} \text{ and } 0 \leq \theta \leq \pi.$$

- **Algebraic definition**

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3.$$

Notice that the dot product of two vectors is a *number*, not a vector.

Why don't we give just one definition of $\vec{v} \cdot \vec{w}$? The reason is that both definitions are equally important; the geometric definition gives us a picture of what the dot product means and the algebraic definition gives us a way of calculating it.

How do we know the two definitions are equivalent—that is, they really do define the same thing? First, we observe that the two definitions give the same result in a particular example. Then we show why they are equivalent in general.



Figure 13.26: The vectors \vec{v} and \vec{w}

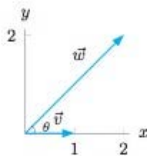


Figure 13.27: Calculating the dot product of the vectors $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$ geometrically and algebraically gives the same result

Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or 45° , and the lengths of the vectors are given by

$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

Thus,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta = 1 \cdot 2\sqrt{2} \cos\left(\frac{\pi}{4}\right) = 2.$$

Using the algebraic definition, we get the same result:

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + 0 \cdot 2 = 2.$$

Why the Two Definitions of the Dot Product Give the Same Result

In the previous example, the two definitions give the same value for the dot product. To show that the geometric and algebraic definitions of the dot product always give the same result, we must show that, for any vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ with an angle θ between them:

$$\|\vec{v}\| \|\vec{w}\| \cos \theta = v_1w_1 + v_2w_2 + v_3w_3.$$

One method follows; a method that does not use trigonometry is given in Problem 69 on page 743.

Using the Law of Cosines. Suppose that $0 < \theta < \pi$, so that the vectors \vec{v} and \vec{w} form a triangle. (See Figure 13.28.) By the Law of Cosines, we have

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\| \|\vec{w}\| \cos \theta.$$

This result is also true for $\theta = 0$ and $\theta = \pi$. We calculate the lengths using components:

$$\begin{aligned} \|\vec{v}\|^2 &= v_1^2 + v_2^2 + v_3^2 \\ \|\vec{w}\|^2 &= w_1^2 + w_2^2 + w_3^2 \\ \|\vec{v} - \vec{w}\|^2 &= (v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2 \\ &= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + v_3^2 - 2v_3w_3 + w_3^2. \end{aligned}$$

Substituting into the Law of Cosines and canceling, we see that

$$-2v_1w_1 - 2v_2w_2 - 2v_3w_3 = -2\|\vec{v}\| \|\vec{w}\| \cos \theta.$$

Therefore we have the result we wanted, namely that:

$$v_1w_1 + v_2w_2 + v_3w_3 = \|\vec{v}\| \|\vec{w}\| \cos \theta.$$

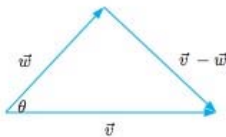


Figure 13.28: Triangle used in the justification of $\|\vec{v}\| \|\vec{w}\| \cos \theta = v_1w_1 + v_2w_2 + v_3w_3$

Properties of the Dot Product

The following properties of the dot product can be justified using the algebraic definition; see Problem 63 on page 742. For a geometric interpretation of Property 3, see Problem 66.

Properties of the Dot Product. For any vectors \vec{u} , \vec{v} , and \vec{w} and any scalar λ ,

- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- $\vec{v} \cdot (\lambda\vec{w}) = \lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w}$
- $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$

Perpendicularity, Magnitude, and Dot Products

Two vectors are perpendicular if the angle between them is $\pi/2$ or 90° . Since $\cos(\pi/2) = 0$, if \vec{v} and \vec{w} are perpendicular, then $\vec{v} \cdot \vec{w} = 0$. Conversely, provided that $\vec{v} \cdot \vec{w} = 0$, then $\cos \theta = 0$, so $\theta = \pi/2$ and the vectors are perpendicular. Thus, we have the following result:

Two non-zero vectors \vec{v} and \vec{w} are **perpendicular**, or **orthogonal**, if and only if

$$\vec{v} \cdot \vec{w} = 0.$$

For example: $\vec{i} \cdot \vec{j} = 0$, $\vec{j} \cdot \vec{k} = 0$, $\vec{i} \cdot \vec{k} = 0$.

If we take the dot product of a vector with itself, then $\theta = 0$ and $\cos \theta = 1$. For any vector \vec{v} :

Magnitude and dot product are related as follows:

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

For example: $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$, $\vec{k} \cdot \vec{k} = 1$.

Using the Dot Product

Depending on the situation, one definition of the dot product may be more convenient to use than the other. In Example 2, the geometric definition is the only one that can be used because we are not given components. In Example 3, the algebraic definition is used.

Example 2 Suppose the vector \vec{b} is fixed and has length 2; the vector \vec{a} is free to rotate and has length 3. What are the maximum and minimum values of the dot product $\vec{a} \cdot \vec{b}$ as the vector \vec{a} rotates through all possible positions? What positions of \vec{a} and \vec{b} lead to these values?

Solution The geometric definition gives $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta = 3 \cdot 2 \cos \theta = 6 \cos \theta$. Thus, the maximum value of $\vec{a} \cdot \vec{b}$ is 6, and it occurs when $\cos \theta = 1$ so $\theta = 0$, that is, when \vec{a} and \vec{b} point in the same direction. The minimum value of $\vec{a} \cdot \vec{b}$ is -6 , and it occurs when $\cos \theta = -1$ so $\theta = \pi$, that is, when \vec{a} and \vec{b} point in opposite directions. (See Figure 13.29.)

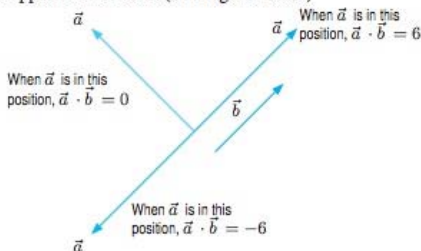


Figure 13.29: Maximum and minimum values of $\vec{a} \cdot \vec{b}$ obtained from a fixed vector \vec{b} of length 2 and rotating vector \vec{a} of length 3

Example 3 Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u} = \vec{i} + \sqrt{3}\vec{k}, \quad \vec{v} = \vec{i} + \sqrt{3}\vec{j}, \quad \vec{w} = \sqrt{3}\vec{i} + \vec{j} - \vec{k}.$$

Solution The geometric definition tells us that two vectors are perpendicular if and only if their dot product is zero. Since the vectors are given in components, we calculate dot products using the algebraic definition:

$$\vec{v} \cdot \vec{u} = (\vec{i} + \sqrt{3}\vec{j} + 0\vec{k}) \cdot (\vec{i} + 0\vec{j} + \sqrt{3}\vec{k}) = 1 \cdot 1 + \sqrt{3} \cdot 0 + 0 \cdot \sqrt{3} = 1,$$

$$\vec{v} \cdot \vec{w} = (\vec{i} + \sqrt{3}\vec{j} + 0\vec{k}) \cdot (\sqrt{3}\vec{i} + \vec{j} - \vec{k}) = 1 \cdot \sqrt{3} + \sqrt{3} \cdot 1 + 0(-1) = 2\sqrt{3},$$

$$\vec{w} \cdot \vec{u} = (\sqrt{3}\vec{i} + \vec{j} - \vec{k}) \cdot (\vec{i} + 0\vec{j} + \sqrt{3}\vec{k}) = \sqrt{3} \cdot 1 + 1 \cdot 0 + (-1) \cdot \sqrt{3} = 0.$$

So the only two vectors that are perpendicular are \vec{w} and \vec{u} .

Example 4 Compute the angle between the vectors \vec{v} and \vec{w} from Example 3.

Solution We know that $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta$, so $\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$. From Example 3, we know that $\vec{v} \cdot \vec{w} = 2\sqrt{3}$. This gives:

$$\cos \theta = \frac{2\sqrt{3}}{\|\vec{v}\| \|\vec{w}\|} = \frac{2\sqrt{3}}{\sqrt{1^2 + (\sqrt{3})^2 + 0^2} \sqrt{(\sqrt{3})^2 + 1^2 + (-1)^2}} = \frac{\sqrt{3}}{\sqrt{5}}$$

so $\theta = \arccos\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 39.2315^\circ$.

Normal Vectors and the Equation of a Plane

In Section 12.4 we wrote the equation of a plane given its x -slope, y -slope and z -intercept. Now we write the equation of a plane using a vector \vec{n} and a point P_0 . The key idea is that all the displacement vectors from P_0 that are perpendicular to \vec{n} form a plane. To picture this, imagine a pencil balanced on a table, with other pencils fanned out on the table in different directions. The upright pencil is \vec{n} , its base is P_0 , the other pencils are perpendicular displacement vectors, and the table is the plane.

More formally, a *normal vector* to a plane is a vector that is perpendicular to the plane, that is, it is perpendicular to every displacement vector between any two points in the plane. Let $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ be a normal vector to the plane, let $P_0 = (x_0, y_0, z_0)$ be a fixed point in the plane, and let $P = (x, y, z)$ be any other point in the plane. Then $\vec{P_0P} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$ is a vector whose head and tail both lie in the plane. (See Figure 13.30.) Thus, the vectors \vec{n} and $\vec{P_0P}$ are perpendicular, so $\vec{n} \cdot \vec{P_0P} = 0$. The algebraic definition of the dot product gives $\vec{n} \cdot \vec{P_0P} = a(x-x_0) + b(y-y_0) + c(z-z_0)$, so we obtain the following result:

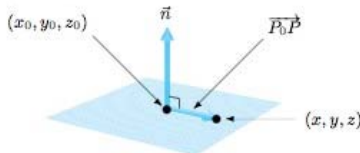


Figure 13.30: Plane with normal \vec{n} and containing a fixed point (x_0, y_0, z_0)

The equation of the plane with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

Letting $d = ax_0 + by_0 + cz_0$ (a constant), we can write the equation of the plane in the form

$$ax + by + cz = d.$$

Example 5 (a) Find the equation of the plane perpendicular to $\vec{n} = -\vec{i} + 3\vec{j} + 2\vec{k}$ and passing through the point $(1, 0, 4)$.
 (b) Find a vector parallel to the plane.

Solution (a) The equation of the plane is

$$-(x-1) + 3(y-0) + 2(z-4) = 0,$$

which can be written as

$$-x + 3y + 2z = 7.$$

- (b) Any vector \vec{v} that is perpendicular to \vec{n} is also parallel to the plane, so we look for any vector satisfying $\vec{v} \cdot \vec{n} = 0$; for example, $\vec{v} = 3\vec{i} + \vec{j}$. There are many other possible vectors.

Example 6 Find a normal vector to the plane with equation (a) $x - y + 2z = 5$ (b) $z = 0.5x + 1.2y$.

- Solution** (a) Since the coefficients of \vec{i} , \vec{j} , and \vec{k} in a normal vector are the coefficients of x , y , and z in the equation of the plane, a normal vector is $\vec{n} = \vec{i} - \vec{j} + 2\vec{k}$.
 (b) Before we can find a normal vector, we rewrite the equation of the plane in the form

$$0.5x + 1.2y - z = 0.$$

Thus, a normal vector is $\vec{n} = 0.5\vec{i} + 1.2\vec{j} - \vec{k}$.

The Dot Product in n Dimensions

The algebraic definition of the dot product can be extended to vectors in higher dimensions.

If $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$ then the dot product of \vec{u} and \vec{v} is the **scalar**

$$\vec{u} \cdot \vec{v} = u_1v_1 + \dots + u_nv_n.$$

Example 7 A video store sells videos, tapes, CDs, and computer games. We define the quantity vector $\vec{q} = (q_1, q_2, q_3, q_4)$, where q_1, q_2, q_3, q_4 denote the quantities sold of each of the items, and the price vector $\vec{p} = (p_1, p_2, p_3, p_4)$, where p_1, p_2, p_3, p_4 denote the price per unit of each item. What does the dot product $\vec{p} \cdot \vec{q}$ represent?

Solution The dot product is $\vec{p} \cdot \vec{q} = p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4$. The quantity p_1q_1 represents the revenue received by the store for the videos, p_2q_2 represents the revenue for the tapes, and so on. The dot product represents the total revenue received by the store for the sale of these four items.

Resolving a Vector into Components: Projections

In Section 13.1, we resolved a vector into components parallel to the axes. Now we see how to resolve a vector, \vec{v} , into components, called $\vec{v}_{\text{parallel}}$ and \vec{v}_{perp} , which are parallel and perpendicular, respectively, to a given non-zero vector, \vec{u} . (See Figure 13.31.)

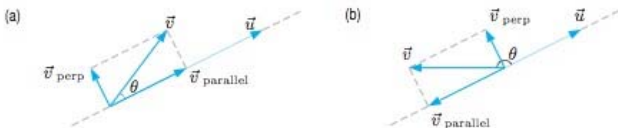


Figure 13.31: Resolving \vec{v} into components parallel and perpendicular to \vec{u}
 (a) $0 < \theta < \pi/2$ (b) $\pi/2 < \theta < \pi$

The projection of \vec{v} on \vec{u} , written $\vec{v}_{\text{parallel}}$, measures (in some sense) how much the vector \vec{v} is aligned with the vector \vec{u} . The length of $\vec{v}_{\text{parallel}}$ is the length of the shadow cast by \vec{v} on a line in the direction of \vec{u} .

To compute $\vec{v}_{\text{parallel}}$, we assume \vec{u} is a unit vector. (If not, create one by dividing by its length.) Then Figure 13.31(a) shows that, if $0 \leq \theta \leq \pi/2$:

$$\|\vec{v}_{\text{parallel}}\| = \|\vec{v}\| \cos \theta = \vec{v} \cdot \vec{u} \quad (\text{since } \|\vec{u}\| = 1).$$

Now $\vec{v}_{\text{parallel}}$ is a scalar multiple of \vec{u} , and since \vec{u} is a unit vector,

$$\vec{v}_{\text{parallel}} = (\|\vec{v}\| \cos \theta) \vec{u} = (\vec{v} \cdot \vec{u}) \vec{u}.$$

A similar argument shows that if $\pi/2 < \theta \leq \pi$, as in Figure 13.31(b), this formula for $\vec{v}_{\text{parallel}}$ still holds. The vector \vec{v}_{perp} is specified by

$$\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

Thus, we have the following results:

Projection of \vec{v} on the Line in the Direction of the Unit Vector \vec{u}

If $\vec{v}_{\text{parallel}}$ and \vec{v}_{perp} are components of \vec{v} that are parallel and perpendicular, respectively, to \vec{u} , then

$$\text{Projection of } \vec{v} \text{ onto } \vec{u} = \vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\vec{u} \quad \text{provided } \|\vec{u}\| = 1$$

$$\text{and } \vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perp}} \quad \text{so } \vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

Example 8

Figure 13.32 shows the force the wind exerts on the sail of a sailboat. Find the component of the force in the direction in which the sailboat is traveling.

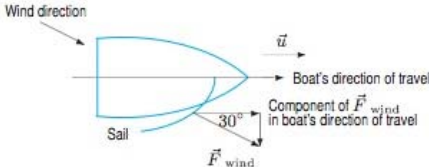


Figure 13.32: Wind moving a sailboat

Solution

Let \vec{u} be a unit vector in the direction of travel. The force of the wind on the sail makes an angle of 30° with \vec{u} . Thus, the component of this force in the direction of \vec{u} is

$$\vec{F}_{\text{parallel}} = (\vec{F} \cdot \vec{u})\vec{u} = \|\vec{F}\|(\cos 30^\circ)\vec{u} = 0.87\|\vec{F}\|\vec{u}.$$

Thus, the boat is being pushed forward with about 87% of the total force due to the wind. (In fact, the interaction of wind and sail is much more complex than this model suggests.)

A Physical Interpretation of the Dot Product: Work

In physics, the word “work” has a different meaning from its everyday meaning. In physics, when a force of magnitude F acts on an object through a distance d , we say the *work*, W , done by the force is

$$W = Fd,$$

provided the force and the displacement are in the same direction. For example, if a 1 kg body falls 10 meters under the force of gravity, which is 9.8 newtons, then the work done by gravity is

$$W = (9.8 \text{ newtons}) \cdot (10 \text{ meters}) = 98 \text{ joules}.$$

What if the force and the displacement are not in the same direction? Suppose a force \vec{F} acts on an object as it moves along a displacement vector \vec{d} . Let θ be the angle between \vec{F} and \vec{d} . First, we assume $0 \leq \theta \leq \pi/2$. Figure 13.33 shows how we can resolve \vec{F} into components that are parallel and perpendicular to \vec{d} :

$$\vec{F} = \vec{F}_{\text{parallel}} + \vec{F}_{\text{perp}}.$$

Then the work done by \vec{F} is defined to be

$$W = \|\vec{F}_{\text{parallel}}\| \|\vec{d}\|.$$

We see from Figure 13.33 that $\vec{F}_{\text{parallel}}$ has magnitude $\|\vec{F}\| \cos \theta$. So the work is given by the dot product:

$$W = (\|\vec{F}\| \cos \theta) \|\vec{d}\| = \|\vec{F}\| \|\vec{d}\| \cos \theta = \vec{F} \cdot \vec{d}.$$

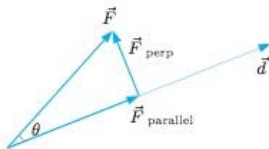


Figure 13.33: Resolving the force \vec{F} into two forces, one parallel to \vec{d} , one perpendicular to \vec{d}

The formula $W = \vec{F} \cdot \vec{d}$ holds when $\pi/2 < \theta \leq \pi$ also. In that case, the work done by the force is negative and the object is moving against the force. Thus, we have the following definition:

The work, W , done by a force \vec{F} acting on an object through a displacement \vec{d} is given by

$$W = \vec{F} \cdot \vec{d}.$$

Example 9 How much work does the wind do on the sailboat from Example 8 if the boat moves 20 m and the wind's force is 120 newtons?

Solution From Example 8, we know that the force of the wind \vec{F} makes a 30° angle with the boat's displacement \vec{d} . Since $\|\vec{F}\| = 120$ and $\|\vec{d}\| = 20$, the work done by the wind on the boat is

$$W = \vec{F} \cdot \vec{d} = \|\vec{F}\| \|\vec{d}\| \cos 30^\circ = 2078.461 \text{ joules.}$$

Notice that if the vectors \vec{F} and \vec{d} are parallel and in the same direction, with magnitudes F and d , then $\cos \theta = \cos 0 = 1$, so $W = \|\vec{F}\| \|\vec{d}\| = Fd$, which is the original definition. When the vectors are perpendicular, $\cos \theta = \cos(\pi/2) = 0$, so $W = 0$ and no work is done in the technical definition of the word. For example, if you carry a heavy box across the room at the same horizontal height, no work is done by gravity because the force of gravity is vertical but the motion is horizontal.

Exercises and Problems for Section 13.3

Exercises

For Exercises 1–9, perform the following operations on the given 3-dimensional vectors.

$$\vec{a} = 2\vec{j} + \vec{k} \quad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k} \quad \vec{c} = \vec{i} + 6\vec{j}$$

$$\vec{y} = 4\vec{i} - 7\vec{j} \quad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}$$

1. $\vec{a} \cdot \vec{y}$

2. $\vec{c} \cdot \vec{y}$

3. $\vec{a} \cdot \vec{b}$

5. $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y}$

7. $(\vec{a} \cdot \vec{b})\vec{a}$

9. $((\vec{c} \cdot \vec{z})\vec{a}) \cdot \vec{a}$

4. $\vec{a} \cdot \vec{z}$

6. $\vec{a} \cdot (\vec{c} + \vec{y})$

8. $(\vec{a} \cdot \vec{y})(\vec{c} \cdot \vec{z})$

In Exercises 10–14, find a normal vector to the plane.

10. $2x + y - z = 5$
11. $2(x - z) = 3(x + y)$
12. $1.5x + 3.2y + z = 0$
13. $z = 3x + 4y - 7$
14. $\pi(x - 1) = (1 - \pi)(y - z) + \pi$

In Exercises 15–21, find an equation of a plane that satisfies the given conditions.

15. Through $(1, 5, 2)$ perpendicular to $3\vec{i} - \vec{j} + 4\vec{k}$
16. Through $(2, -1, 3)$ perpendicular to $5\vec{i} + 4\vec{j} - \vec{k}$.
17. Through $(1, 3, 5)$ and normal to $\vec{i} - \vec{j} + \vec{k}$.

18. Perpendicular to $5\vec{i} + \vec{j} - 2\vec{k}$ and passing through $(0, 1, -1)$.
19. Parallel to $2x + 4y - 3z = 1$ and through $(1, 0, -1)$.
20. Through $(-2, 3, 2)$ and parallel to $3x + y + z = 4$.
21. Perpendicular to $\vec{v} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and through $(4, 5, -2)$.

In Exercises 22–26, compute the angle between the vectors.

22. $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.
23. $\vec{i} + \vec{k}$ and $\vec{j} - \vec{k}$.
24. $\vec{i} + \vec{j} - \vec{k}$ and $2\vec{i} + 3\vec{j} + \vec{k}$.
25. $\vec{i} + \vec{j}$ and $\vec{i} + 2\vec{j} - \vec{k}$.
26. \vec{i} and $2\vec{i} + 3\vec{j} - \vec{k}$.

Problems

27. Give a unit vector

- (a) In the same direction as $\vec{v} = 2\vec{i} + 3\vec{j}$.
- (b) Perpendicular to \vec{v} .

28. A plane has equation $z = 5x - 2y + 7$.

- (a) Find a value of λ making the vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ normal to the plane.
- (b) Find a value of a so that the point $(a + 1, a, a - 1)$ lies on the plane.

29. Consider the plane $5x - y + 7z = 21$.

- (a) Find a point on the x -axis on this plane.
- (b) Find two other points on the plane.
- (c) Find a vector perpendicular to the plane.
- (d) Find a vector parallel to the plane.

30. (a) Find a vector perpendicular to the plane $z = 2 + 3x - y$.

- (b) Find a vector parallel to the plane.

31. (a) Find a vector perpendicular to the plane $z = 2x + 3y$.

- (b) Find a vector parallel to the plane.

32. Match the planes in (a)–(d) with one or more of the descriptions in (I)–(IV). No reasons needed.

- | | |
|----------------------|---------------------------|
| (a) $3x - y + z = 0$ | (b) $4x + y + 2z - 5 = 0$ |
| (c) $x + y = 5$ | (d) $x = 5$ |

I Goes through the origin.

II Has a normal vector parallel to the xy -plane.

III Goes through the point $(0, 5, 0)$.

IV Has a normal vector whose dot products with \vec{i} , \vec{j} , \vec{k} are all positive.

33. Which pairs (if any) of vectors from the following list

- (a) Are perpendicular?
- (b) Are parallel?
- (c) Have an angle less than $\pi/2$ between them?

(d) Have an angle of more than $\pi/2$ between them?

$$\vec{a} = \vec{i} - 3\vec{j} - \vec{k}, \quad \vec{b} = \vec{i} + \vec{j} + 2\vec{k},$$

$$\vec{c} = -2\vec{i} - \vec{j} + \vec{k}, \quad \vec{d} = -\vec{i} - \vec{j} + \vec{k}.$$

34. List any vectors that are parallel to each other and any vectors that are perpendicular to each other:

$\vec{v}_1 = \vec{i} - 2\vec{j}$	$\vec{v}_2 = 2\vec{i} + 4\vec{j}$
$\vec{v}_3 = 3\vec{i} + 1.5\vec{j}$	$\vec{v}_4 = -1.2\vec{i} + 2.4\vec{j}$
$\vec{v}_5 = -5\vec{i} - 2.5\vec{j}$	$\vec{v}_6 = 12\vec{i} - 12\vec{j}$
$\vec{v}_7 = 4\vec{i} + 2\vec{j}$	$\vec{v}_8 = 3\vec{i} - 6\vec{j}$
$\vec{v}_9 = 0.70\vec{i} - 0.35\vec{j}$	

35. (a) Give a vector that is parallel to, but not equal to, $\vec{v} = 4\vec{i} + 3\vec{j}$.

- (b) Give a vector that is perpendicular to \vec{v} .

36. For what values of t are $\vec{u} = t\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = t\vec{i} + t\vec{j} - 2\vec{k}$ perpendicular? Are there values of t for which \vec{u} and \vec{v} are parallel?

37. Let θ be the angle between \vec{v} and \vec{w} , with $0 < \theta < \pi/2$. What is the effect on $\vec{v} \cdot \vec{w}$ of increasing each of the following quantities? Does $\vec{v} \cdot \vec{w}$ increase or decrease?

- | | |
|-------------------|--------------|
| (a) $\ \vec{v}\ $ | (b) θ |
|-------------------|--------------|

38. Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular, to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

39. Find angle BAC if $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

40. The points $(5, 0, 0)$, $(0, -3, 0)$, and $(0, 0, 2)$ form a triangle. Find the lengths of the sides of the triangle and each of its angles.

41. Let S be the triangle with vertices $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$.

- (a) Find the length of the shortest side of S .
 (b) Find the cosine of the angle BAC at vertex A .

In Problems 42–47, given $\vec{v} = 3\vec{i} + 4\vec{j}$ and force vector \vec{F} , find:

- (a) The component of \vec{F} parallel to \vec{v} .
 (b) The component of \vec{F} perpendicular to \vec{v} .
 (c) The work, W , done by force \vec{F} through displacement \vec{v} .

42. $\vec{F} = 4\vec{i} + \vec{j}$

43. $\vec{F} = 0.2\vec{i} - 0.5\vec{j}$

44. $\vec{F} = 9\vec{i} + 12\vec{j}$

45. $\vec{F} = -0.4\vec{i} + 0.3\vec{j}$

46. $\vec{F} = -3\vec{i} - 5\vec{j}$

47. $\vec{F} = -6\vec{i} - 8\vec{j}$

In Problems 48–51, the force on an object is $\vec{F} = -20\vec{j}$. For vector \vec{v} , find:

- (a) The component of \vec{F} parallel to \vec{v} .
 (b) The component of \vec{F} perpendicular to \vec{v} .
 (c) The work, W , done by force \vec{F} through displacement \vec{v} .

48. $\vec{v} = 2\vec{i} + 3\vec{j}$

49. $\vec{v} = 5\vec{i} - \vec{j}$

50. $\vec{v} = 3\vec{j}$

51. $\vec{v} = 5\vec{i}$

52. A basketball gymnasium is 25 meters high, 80 meters wide and 200 meters long. For a half-time stunt, the cheerleaders want to run two strings, one from each of the two corners above one basket to the diagonally opposite corners of the gym floor. What is the cosine of the angle made by the strings as they cross?

53. A 100-meter dash is run on a track in the direction of the vector $\vec{v} = 2\vec{i} + 6\vec{j}$. The wind velocity \vec{w} is $5\vec{i} + \vec{j}$ km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? Justify your answer.

54. An airplane is flying toward the southeast. Which of the following wind velocity vectors increases the plane's speed the most? Which slows down the plane the most?

$$\vec{w}_1 = -4\vec{i} - \vec{j} \quad \vec{w}_2 = \vec{i} - 2\vec{j} \quad \vec{w}_3 = -\vec{i} + 8\vec{j}$$

$$\vec{w}_4 = 10\vec{i} + 2\vec{j} \quad \vec{w}_5 = 5\vec{i} - 2\vec{j}$$

55. A canoe is moving with velocity $\vec{v} = 5\vec{i} + 3\vec{j}$ m/sec relative to the water. The velocity of the current in the water is $\vec{c} = \vec{i} + 2\vec{j}$ m/sec.

- (a) What is the speed of the current?
 (b) What is the speed of the current in the direction of the canoe's motion?

56. Find a vector that bisects the smaller of the two angles formed by $3\vec{i} + 4\vec{j}$ and $5\vec{i} - 12\vec{j}$.

57. Find the shortest distance between the planes $2x - 5y + z = 10$ and $z = 5y - 2x$.

58. A street vendor sells six items, with prices p_1 dollars per unit, p_2 dollars per unit, and so on. The vendor's price vector is $\vec{p} = (p_1, p_2, p_3, p_4, p_5, p_6) = (1.00, 3.50, 4.00, 2.75, 5.00, 3.00)$. The vendor sells q_1 units of the first item, q_2 units of the second item, and so on. The vendor's quantity vector is $\vec{q} = (q_1, q_2, q_3, q_4, q_5, q_6) = (43, 57, 12, 78, 20, 35)$. Find $\vec{p} \cdot \vec{q}$, give its units, and explain its significance to the vendor.

59. A course has four exams, weighted 10%, 15%, 25%, 50%, respectively. The class average on each of these exams is 75%, 91%, 84%, 87%, respectively. What do the vectors $\vec{a} = (0.75, 0.91, 0.84, 0.87)$ and $\vec{w} = (0.1, 0.15, 0.25, 0.5)$ represent, in terms of the course? Calculate the dot product $\vec{w} \cdot \vec{a}$. What does it represent, in terms of the course?

60. A consumption vector of three goods is defined by $\vec{x} = (x_1, x_2, x_3)$, where x_1 , x_2 and x_3 are the quantities consumed of the three goods. A budget constraint is represented by the equation $\vec{p} \cdot \vec{x} = k$, where \vec{p} is the price vector of the three goods and k is a constant. Show that the difference between two consumption vectors corresponding to points satisfying the same budget constraint is perpendicular to the price vector \vec{p} .

61. What does Property 2 of the dot product in the box on page 735 say geometrically?

62. Show that the vectors $(\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ and \vec{c} are perpendicular.

63. Show why each of the properties of the dot product in the box on page 735 follows from the algebraic definition of the dot product:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

64. Show that if \vec{u} and \vec{v} are two vectors such that

$$\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w}$$

for every vector \vec{w} , then

$$\vec{u} = \vec{v}.$$

65. Show that

$$\frac{\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{v}}{\|\vec{v}\|^2} \quad \text{and} \quad \frac{\vec{u}}{\|\vec{u}\|\|\vec{v}\|} - \frac{\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

have the same magnitude where \vec{u} and \vec{v} are nonzero vectors.

66. Figure 13.34 shows that, given three vectors \vec{u} , \vec{v} , and \vec{w} , the sum of the components of \vec{v} and \vec{w} in the direction of \vec{u} is the component of $\vec{v} + \vec{w}$ in the direction of \vec{u} . (Although the figure is drawn in two dimensions, this result is also true in three dimensions.) Use this figure to explain why the geometric definition of the dot product satisfies $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$.

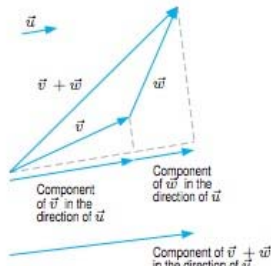


Figure 13.34: Component of $\vec{v} + \vec{w}$ in the direction of \vec{u} is the sum of the components of \vec{v} and \vec{w} in that direction

67. (a) Using the geometric definition of the dot product, show that
- $$\vec{u} \cdot (-\vec{v}) = -(\vec{u} \cdot \vec{v}).$$
- [Hint: What happens to the angle when you multiply \vec{v} by -1 ?]
- (b) Using the geometric definition of the dot product, show that for any negative scalar λ
- $$\vec{u} \cdot (\lambda \vec{v}) = \lambda(\vec{u} \cdot \vec{v})$$
- $$(\lambda \vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{v}).$$
68. The Law of Cosines for a triangle with side lengths a , b , and c , and with angle C opposite side c , says
- $$c^2 = a^2 + b^2 - 2ab \cos C.$$
- On page 735, we used the Law of Cosines to show that the two definitions of the dot product are equivalent. In this problem, use the geometric definition of the dot product and its properties in the box on page 735 to prove the Law of Cosines. [Hint: Let \vec{u} and \vec{v} be the displacement vectors from C to the other two vertices, and express c^2 in terms of \vec{u} and \vec{v} .]
69. Use Problems 66 and 67 and the following steps to show (without trigonometry) that the geometric and algebraic definitions of the dot product are equivalent. Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be any vectors. Write $(\vec{u} \cdot \vec{v})_{\text{geom}}$ for the result of the dot product computed geometrically. Substitute $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and use Problems 66–67 to expand $(\vec{u} \cdot \vec{v})_{\text{geom}}$. Substitute for \vec{v} and expand. Then calculate the dot products $\vec{i} \cdot \vec{i}$, $\vec{i} \cdot \vec{j}$, etc. geometrically.
70. For any vectors \vec{v} and \vec{w} , consider the following function of t :
- $$q(t) = (\vec{v} + t\vec{w}) \cdot (\vec{v} + t\vec{w}).$$
- (a) Explain why $q(t) \geq 0$ for all real t .
- (b) Expand $q(t)$ as a quadratic polynomial in t using the properties on page 735.
- (c) Using the discriminant of the quadratic, show that
- $$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\|.$$

Strengthen Your Understanding

In Problems 71–73, explain what is wrong with the statement.

71. For any 3-dimensional vectors \vec{u} , \vec{v} , \vec{w} , we have $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \cdot \vec{w})$.
72. If $\vec{u} = \vec{i} + \vec{j}$ and $\vec{v} = 2\vec{i} + \vec{j}$, then the component of \vec{v} parallel to \vec{u} is $(\vec{v} \cdot \vec{u})\vec{u} = 3\vec{i} + 3\vec{j}$.
73. A normal vector for the plane $z = 2x + 3y$ is $2\vec{i} + 3\vec{j}$.

In Problems 74–75, give an example of:

74. A point (a, b) such that the displacement vector from $(1, 1)$ to (a, b) is perpendicular to $\vec{i} + 2\vec{j}$.
75. A linear function $f(x, y) = mx + ny + c$ whose graph is perpendicular to $\vec{i} + 2\vec{j} + 3\vec{k}$.

Are the statements in Problems 76–85 true or false? Give reasons for your answer.

76. The quantity $\vec{u} \cdot \vec{v}$ is a vector.

77. The plane $x + 2y - 3z = 5$ has normal vector $\vec{i} + 2\vec{j} - 3\vec{k}$.
78. If $\vec{u} \cdot \vec{v} < 0$ then the angle between \vec{u} and \vec{v} is greater than $\pi/2$.
79. An equation of the plane with normal vector $\vec{i} + \vec{j} + \vec{k}$ containing the point $(1, 2, 3)$ is $z = x + y$.
80. The triangle in 3-space with vertices $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 1, 1)$ has a right angle.
81. The dot product $\vec{v} \cdot \vec{v}$ is never negative.
82. If $\vec{u} \cdot \vec{v} = 0$ then either $\vec{u} = 0$ or $\vec{v} = 0$.
83. If \vec{u} , \vec{v} and \vec{w} are all nonzero, and $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{v} = \vec{w}$.
84. For any vectors \vec{u} and \vec{v} : $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.
85. If $\|\vec{u}\| = 1$, then the vector $\vec{v} - (\vec{v} \cdot \vec{u})\vec{u}$ is perpendicular to \vec{u} .

13.4 THE CROSS PRODUCT

In the previous section we combined two vectors to get a number, the dot product. In this section we see another way of combining two vectors, this time to get a vector, the *cross product*. Any two vectors in 3-space form a parallelogram. We define the cross product using this parallelogram.

The Area of a Parallelogram

Consider the parallelogram formed by the vectors \vec{v} and \vec{w} with an angle of θ between them. Then Figure 13.35 shows

$$\text{Area of parallelogram} = \text{Base} \cdot \text{Height} = \|\vec{v}\| \|\vec{w}\| \sin \theta.$$

How would we compute the area of the parallelogram if we were given \vec{v} and \vec{w} in components, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$? Project 1 on page 755 shows that if \vec{v} and \vec{w} are in the xy -plane so that $v_3 = w_3 = 0$, then

$$\text{Area of parallelogram} = |v_1w_2 - v_2w_1|.$$

What if \vec{v} and \vec{w} do not lie in the xy -plane? The cross product will enable us to compute the area of the parallelogram formed by any two vectors.

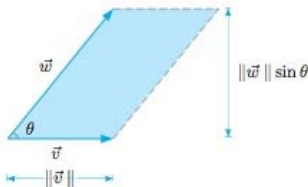


Figure 13.35: Parallelogram formed by \vec{v} and \vec{w} has
Area = $\|\vec{v}\| \|\vec{w}\| \sin \theta$

Definition of the Cross Product

We define the cross product of the vectors \vec{v} and \vec{w} , written $\vec{v} \times \vec{w}$, to be a vector perpendicular to both \vec{v} and \vec{w} . The magnitude of this vector is the area of the parallelogram formed by the two vectors. The direction of $\vec{v} \times \vec{w}$ is given by the normal vector, \vec{n} , to the plane defined by \vec{v} and \vec{w} . If we require that \vec{n} be a unit vector, there are two choices for \vec{n} , pointing out of the plane in opposite directions. We pick one by the following rule (see Figure 13.36):

The right-hand rule: Place \vec{v} and \vec{w} so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from \vec{v} to \vec{w} ; your thumb points in the direction of the normal vector, \vec{n} .

Like the dot product, there are two equivalent definitions of the cross product:

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

- **Geometric definition**

If \vec{v} and \vec{w} are not parallel, then

$$\vec{v} \times \vec{w} = \left(\begin{array}{l} \text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{array} \right) \vec{n} = (\|\vec{v}\| \|\vec{w}\| \sin \theta) \vec{n},$$

where $0 \leq \theta \leq \pi$ is the angle between \vec{v} and \vec{w} and \vec{n} is the unit vector perpendicular to \vec{v} and \vec{w} pointing in the direction given by the right-hand rule. If \vec{v} and \vec{w} are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- **Algebraic definition**

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2) \vec{i} + (v_3 w_1 - v_1 w_3) \vec{j} + (v_1 w_2 - v_2 w_1) \vec{k}$$

where $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$ and $\vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$.

Problems 45 and 48 at the end of this section show that the geometric and algebraic definitions of the cross product give the same result.

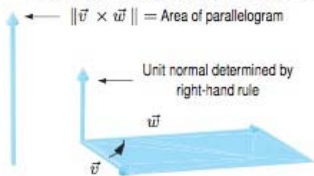


Figure 13.36: Area of parallelogram = $\|\vec{v} \times \vec{w}\|$

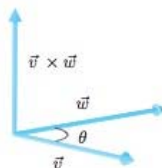


Figure 13.37: The cross product $\vec{v} \times \vec{w}$

The geometric definition shows us that the cross product is *rotation invariant*. Imagine the two vectors \vec{v} and \vec{w} as two metal rods welded together. Attach a third rod whose direction and length correspond to $\vec{v} \times \vec{w}$. (See Figure 13.37.) Then, no matter how we turn this set of rods, the third will still be the cross product of the first two.

The algebraic definition is more easily remembered by writing it as a 3×3 determinant. (See Appendix E.)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2) \vec{i} + (v_3 w_1 - v_1 w_3) \vec{j} + (v_1 w_2 - v_2 w_1) \vec{k}.$$

Example 1 Find $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$.

Solution The vectors \vec{i} and \vec{j} both have magnitude 1 and the angle between them is $\pi/2$. By the right-hand rule, the vector $\vec{i} \times \vec{j}$ is in the direction of \vec{k} , so $\vec{n} = \vec{k}$ and we have

$$\vec{i} \times \vec{j} = (\|\vec{i}\| \|\vec{j}\| \sin \frac{\pi}{2}) \vec{k} = \vec{k}.$$

Similarly, the right-hand rule says that the direction of $\vec{j} \times \vec{i}$ is $-\vec{k}$, so

$$\vec{j} \times \vec{i} = (\|\vec{j}\| \|\vec{i}\| \sin \frac{\pi}{2}) (-\vec{k}) = -\vec{k}.$$

Similar calculations show that $\vec{j} \times \vec{k} = \vec{i}$ and $\vec{k} \times \vec{i} = \vec{j}$.

Example 2 For any vector \vec{v} , find $\vec{v} \times \vec{v}$.

Solution Since \vec{v} is parallel to itself, $\vec{v} \times \vec{v} = \vec{0}$.

Example 3 Find the cross product of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = 3\vec{i} + \vec{k}$ and check that the cross product is perpendicular to both \vec{v} and \vec{w} .

Solution Writing $\vec{v} \times \vec{w}$ as a determinant and expanding it into three two-by-two determinants, we have

$$\begin{aligned}\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\ &= \vec{i}(1(1) - 0(-2)) - \vec{j}(2(1) - 3(-2)) + \vec{k}(2(0) - 3(1)) \\ &= \vec{i} - 8\vec{j} - 3\vec{k}.\end{aligned}$$

To check that $\vec{v} \times \vec{w}$ is perpendicular to \vec{v} , we compute the dot product:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = (2\vec{i} + \vec{j} - 2\vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 2 - 8 + 6 = 0.$$

Similarly,

$$\vec{w} \cdot (\vec{v} \times \vec{w}) = (3\vec{i} + 0\vec{j} + \vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 3 + 0 - 3 = 0.$$

Thus, $\vec{v} \times \vec{w}$ is perpendicular to both \vec{v} and \vec{w} .

Properties of the Cross Product

The right-hand rule tells us that $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ point in opposite directions. The magnitudes of $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$ are the same, so $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$. (See Figure 13.38.)

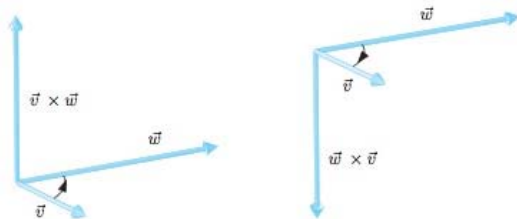


Figure 13.38: Diagram showing $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

This explains the first of the following properties. The other two are derived in Problems 39, 40, and 48 at the end of this section.

Properties of the Cross Product

For vectors \vec{u} , \vec{v} , \vec{w} and scalar λ

- $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$
- $(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w})$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

The Equation of a Plane Through Three Points

As we saw on page 737, the equation of a plane is determined by a point $P_0 = (x_0, y_0, z_0)$ on the plane, and a normal vector, $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

However, a plane can also be determined by three points on it (provided they do not lie on the same line). In that case we can find an equation of the plane by first determining two vectors in the plane and then finding a normal vector using the cross product, as in the following example.

Example 4 Find an equation of the plane containing the points $P = (1, 3, 0)$, $Q = (3, 4, -3)$, and $R = (3, 6, 2)$.

Solution Since the points P and Q are in the plane, the displacement vector between them, \overrightarrow{PQ} , is in the plane, where

$$\overrightarrow{PQ} = (3 - 1)\vec{i} + (4 - 3)\vec{j} + (-3 - 0)\vec{k} = 2\vec{i} + \vec{j} - 3\vec{k}.$$

The displacement vector \overrightarrow{PR} is also in the plane, where

$$\overrightarrow{PR} = (3 - 1)\vec{i} + (6 - 3)\vec{j} + (2 - 0)\vec{k} = 2\vec{i} + 3\vec{j} + 2\vec{k}.$$

Thus, a normal vector, \vec{n} , to the plane is given by

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = 11\vec{i} - 10\vec{j} + 4\vec{k}.$$

Since the point $(1, 3, 0)$ is on the plane, the equation of the plane is

$$11(x - 1) - 10(y - 3) + 4(z - 0) = 0,$$

which simplifies to

$$11x - 10y + 4z = -19.$$

You should check that P , Q , and R satisfy this equation, since they lie on the plane.

Areas and Volumes Using the Cross Product and Determinants

We can use the cross product to calculate the area of the parallelogram with sides \vec{v} and \vec{w} . We say that $\vec{v} \times \vec{w}$ is the *area vector* of the parallelogram. The geometric definition of the cross product tells us that $\vec{v} \times \vec{w}$ is normal to the parallelogram and gives us the following result:

Area of a parallelogram with edges $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ is given by

$$\text{Area} = \|\vec{v} \times \vec{w}\|, \quad \text{where} \quad \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example 5 Find the area of the parallelogram with edges $\vec{v} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{w} = \vec{i} + 3\vec{j} + 2\vec{k}$.

Solution We calculate the cross product:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2+9)\vec{i} - (4+3)\vec{j} + (6-1)\vec{k} = 11\vec{i} - 7\vec{j} + 5\vec{k}.$$

The area of the parallelogram with edges \vec{v} and \vec{w} is the magnitude of the vector $\vec{v} \times \vec{w}$:

$$\text{Area} = \|\vec{v} \times \vec{w}\| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}.$$

Volume of a Parallelepiped

Consider the parallelepiped with sides formed by \vec{a} , \vec{b} , and \vec{c} . (See Figure 13.39.) Since the base is formed by the vectors \vec{b} and \vec{c} , we have

$$\text{Area of base of parallelepiped} = \|\vec{b} \times \vec{c}\|.$$

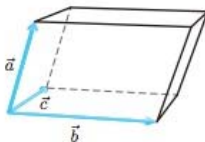


Figure 13.39: Volume of a parallelepiped

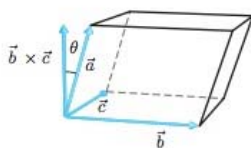


Figure 13.40: The vectors \vec{a} , \vec{b} , \vec{c} are called a right-handed set

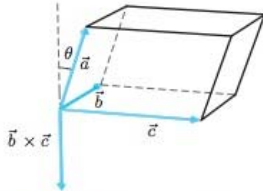


Figure 13.41: The vectors \vec{a} , \vec{b} , \vec{c} are called a left-handed set

The vectors \vec{a} , \vec{b} , and \vec{c} can be arranged either as in Figure 13.40 or as in Figure 13.41. In either case,

$$\text{Height of parallelepiped} = \|\vec{a}\| \cos \theta,$$

where θ is the angle shown in the figures. In Figure 13.40 the angle θ is less than $\pi/2$, so the product, $(\vec{b} \times \vec{c}) \cdot \vec{a}$, called the *triple product*, is positive. Thus, in this case

$$\text{Volume of parallelepiped} = \text{Base} \cdot \text{Height} = \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos \theta = (\vec{b} \times \vec{c}) \cdot \vec{a}.$$

In Figure 13.41, the angle, $\pi - \theta$, between \vec{a} and $\vec{b} \times \vec{c}$ is more than $\pi/2$, so the product $(\vec{b} \times \vec{c}) \cdot \vec{a}$ is negative. Thus, in this case we have

$$\begin{aligned} \text{Volume} &= \text{Base} \cdot \text{Height} = \|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos \theta = -\|\vec{b} \times \vec{c}\| \cdot \|\vec{a}\| \cos(\pi - \theta) \\ &= -(\vec{b} \times \vec{c}) \cdot \vec{a} = \left| (\vec{b} \times \vec{c}) \cdot \vec{a} \right|. \end{aligned}$$

Therefore, in both cases the volume is given by $\left| (\vec{b} \times \vec{c}) \cdot \vec{a} \right|$. Using determinants, we can write

Volume of a parallelepiped with edges \vec{a} , \vec{b} , \vec{c} is given by

$$\text{Volume} = \left| (\vec{b} \times \vec{c}) \cdot \vec{a} \right| = \text{Absolute value of the determinant} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Exercises and Problems for Section 13.4

Exercises

In Exercises 1–7, use the algebraic definition to find $\vec{v} \times \vec{w}$.

- $\vec{v} = \vec{k}, \vec{w} = \vec{j}$
- $\vec{v} = -\vec{i}, \vec{w} = \vec{j} + \vec{k}$
- $\vec{v} = \vec{i} + \vec{k}, \vec{w} = \vec{i} + \vec{j}$
- $\vec{v} = \vec{i} + \vec{j} + \vec{k}, \vec{w} = \vec{i} + \vec{j} + \vec{k}$
- $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}, \vec{w} = \vec{i} + 2\vec{j} - \vec{k}$
- $\vec{v} = 2\vec{i} - \vec{j} - \vec{k}, \vec{w} = -6\vec{i} + 3\vec{j} + 3\vec{k}$
- $\vec{v} = -3\vec{i} + 5\vec{j} + 4\vec{k}, \vec{w} = \vec{i} - 3\vec{j} - \vec{k}$

Use the geometric definition in Exercises 8–9 to find:

- $2\vec{i} \times (\vec{i} + \vec{j})$
- $(\vec{i} + \vec{j}) \times (\vec{i} - \vec{j})$

In Exercises 10–11, use the properties on page 746 to find:

- $((\vec{i} + \vec{j}) \times \vec{i}) \times \vec{j}$
- $(\vec{i} + \vec{j}) \times (\vec{i} \times \vec{j})$

- For $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$, find $\vec{a} \times \vec{b}$ and check that it is perpendicular to both \vec{a} and \vec{b} .
- If $\vec{v} = 3\vec{i} - 2\vec{j} + 4\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$, find $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$. What is the relation between the two answers?

In Exercises 14–15, find an equation for the plane through the points.

- $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.
- $(3, 4, 2), (-2, 1, 0), (0, 2, 1)$.

In Exercises 16–19, find the volume of the parallelogram with edges $\vec{a}, \vec{b}, \vec{c}$.

- $\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}, \vec{b} = 5\vec{i} + 4\vec{j} + 3\vec{k}, \vec{c} = \vec{i} + \vec{j} + \vec{k}$.
- $\vec{a} = -\vec{i} + \vec{j} + \vec{k}, \vec{b} = \vec{i} - \vec{j} + \vec{k}, \vec{c} = \vec{i} + \vec{j} - \vec{k}$.
- $\vec{a} = -\vec{i} + 8\vec{j} + 7\vec{k}, \vec{b} = 2\vec{j} + 9\vec{k}, \vec{c} = 3\vec{k}$.
- $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}, \vec{b} = \vec{i} + \vec{k}, \vec{c} = \vec{j} + \vec{k}$.

Problems

- Find a vector parallel to the line of intersection of the planes given by the equations $2x - 3y + 5z = 2$ and $4x + y - 3z = 7$.
- Find the equation of the plane through the origin that is perpendicular to the line of intersection of the planes in Problem 20.
- Find the equation of the plane through the point $(4, 5, 6)$ and perpendicular to the line of intersection of the planes in Problem 20.
- Find an equation for the plane through the origin containing the points $(1, 3, 0)$ and $(2, 4, 1)$.
- Find a vector parallel to the line of intersection of the two planes $4x - 3y + 2z = 12$ and $x + 5y - z = 25$.
- Find a vector parallel to the intersection of the planes $2x - 3y + 5z = 2$ and $4x + y - 3z = 7$.
- Find the equation of the plane through the origin that is perpendicular to the line of intersection of the planes in Problem 25.
- Find the equation of the plane through the point $(4, 5, 6)$ that is perpendicular to the line of intersection of the planes in Problem 25.
- Find the equation of a plane through the origin and perpendicular to $x - y + z = 5$ and $2x + y - 2z = 7$.
- Given the points $P = (1, 2, 3)$, $Q = (3, 5, 7)$, and $R = (2, 5, 3)$, find:
 - A unit vector perpendicular to a plane containing P, Q, R .
 - The angle between PQ and PR .
 - The area of the triangle PQR .
 - The distance from R to the line through P and Q .
- Let $A = (-1, 3, 0)$, $B = (3, 2, 4)$, and $C = (1, -1, 5)$.
 - Find an equation for the plane that passes through these three points.
 - Find the area of the triangle determined by these three points.
- If \vec{v} and \vec{w} are both parallel to the xy -plane, what can you conclude about $\vec{v} \times \vec{w}$? Explain.
- Suppose $\vec{v} \cdot \vec{w} = 5$ and $\|\vec{v} \times \vec{w}\| = 3$, and the angle between \vec{v} and \vec{w} is θ . Find
 - $\tan \theta$
 - θ .
- If $\vec{v} \times \vec{w} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, and $\vec{v} \cdot \vec{w} = 3$, find $\tan \theta$ where θ is the angle between \vec{v} and \vec{w} .
- Suppose $\vec{v} \cdot \vec{w} = 8$ and $\vec{v} \times \vec{w} = 12\vec{i} - 3\vec{j} + 4\vec{k}$ and that the angle between \vec{v} and \vec{w} is θ . Find
 - $\tan \theta$
 - θ
- Why does a baseball curve? The baseball in Figure 13.42 has velocity \vec{v} meters/sec and is spinning at ω radians per second about an axis in the direction of the unit vector \vec{n} . The ball experiences a force, called the Magnus force,² \vec{F}_M , that is proportional to $\omega \vec{n} \times \vec{v}$.

²Named after German physicist Heinrich Magnus, who first described it in 1853.

- (a) What is the effect on \vec{F}_M of increasing ω ?
 (b) The ball in Figure 13.42 is moving away from you. What is the direction of the Magnus force?



Figure 13.42: Spinning baseball

36. The point P in Figure 13.43 has position vector \vec{v} obtained by rotating the position vector \vec{r} of the point (x, y) by 90° counterclockwise about the origin.
 (a) Use the geometric definition of the cross product to explain why $\vec{v} = \vec{k} \times \vec{r}$.
 (b) Find the coordinates of P .

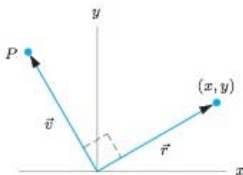


Figure 13.43

37. The points $P_1 = (0, 0, 0)$, $P_2 = (2, 4, 2)$, $P_3 = (3, 0, 0)$, and $P_4 = (5, 4, 2)$ are vertices of a parallelogram.
 (a) Find the displacement vectors along each of the four sides. Check that these are equal in pairs.
 (b) Find the area of the parallelogram.
 38. Using the parallelogram in Problem 37 as a base, create a parallelepiped with side $\vec{P}_1\vec{P}_5$ where $P_5 = (1, 0, 4)$. Find the volume of this parallelepiped.
 39. Use the algebraic definition to check that

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}).$$

40. If \vec{v} and \vec{w} are non-zero vectors, use the geometric definition of the cross product to explain why

$$(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w}).$$

Consider the cases $\lambda > 0$, and $\lambda = 0$, and $\lambda < 0$ separately.

41. Use a parallelepiped to show that $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ for any vectors \vec{a} , \vec{b} , and \vec{c} .

42. Show that $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$.

43. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

Geometrically, what does this imply about \vec{a} , \vec{b} , and \vec{c} ?

44. If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ and $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ are any three vectors in space, show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

45. Use the fact that $\vec{i} \times \vec{i} = \vec{0}$, $\vec{i} \times \vec{j} = \vec{k}$, $\vec{i} \times \vec{k} = -\vec{j}$, and so on, together with the properties on page 746 to derive the algebraic definition for the cross product.
 46. In this problem, we arrive at the algebraic definition for the cross product by a different route. Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$. We seek a vector $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$ that is perpendicular to both \vec{a} and \vec{b} . Use this requirement to construct two equations for x , y , and z . Eliminate x and solve for y in terms of z . Then eliminate y and solve for x in terms of z . Since z can be any value whatsoever (the direction of \vec{v} is unaffected), select the value for z which eliminates the denominator in the equation you obtained. How does the resulting expression for \vec{v} compare to the formula we derived on page 745?
 47. For vectors \vec{a} and \vec{b} , let $\vec{c} = \vec{a} \times (\vec{b} \times \vec{a})$.

- (a) Show that \vec{c} lies in the plane containing \vec{a} and \vec{b} .
 (b) Use Problems 41 and 42 to show that $\vec{a} \cdot \vec{c} = 0$ and $\vec{b} \cdot \vec{c} = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$.
 (c) Show that

$$\vec{a} \times (\vec{b} \times \vec{a}) = \|\vec{a}\|^2 \vec{b} - (\vec{a} \cdot \vec{b}) \vec{a}.$$

48. Use the result of Problem 41 to show that the cross product distributes over addition. First, use distributivity for the dot product to show that for any vector \vec{d} ,

$$[(\vec{a} + \vec{b}) \times \vec{c}] \cdot \vec{d} = [(\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})] \cdot \vec{d}.$$

Next, show that for any vector \vec{d} ,

$$[[(\vec{a} + \vec{b}) \times \vec{c}] - (\vec{a} \times \vec{c}) - (\vec{b} \times \vec{c})] \cdot \vec{d} = 0.$$

Finally, explain why you can conclude that

$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}).$$

49. Figure 13.44 shows the tetrahedron determined by three vectors \vec{a} , \vec{b} , \vec{c} . The area vector of a face is a vector perpendicular to the face, pointing outward, whose magnitude is the area of the face. Show that the sum of the four outward pointing area vectors of the faces equals the zero vector.

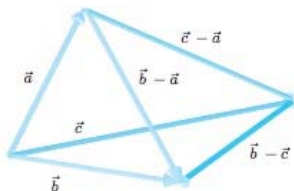


Figure 13.44

In Problems 50–52, find the vector representing the area of a surface. The magnitude of the vector equals the magnitude of the area; the direction is perpendicular to the surface. Since there are two perpendicular directions, we pick one by giving an orientation for the surface.

50. The rectangle with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(2, 1, 0)$, and $(2, 0, 0)$, oriented so that it faces downward.
51. The circle of radius 2 in the yz -plane, facing in the direction of the positive x -axis.
52. The triangle ABC , oriented upward, where $A = (1, 2, 3)$, $B = (3, 1, 2)$, and $C = (2, 1, 3)$.
53. This problem relates the area of a parallelogram S lying in the plane $z = mx + ny + c$ to the area of its projection

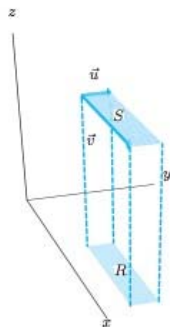


Figure 13.45

R in the xy -plane. Let S be determined by the vectors $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. See Figure 13.45.

- (a) Find the area of S .
 (b) Find the area of R .
 (c) Find m and n in terms of the components of \vec{u} and \vec{v} .
 (d) Show that

$$\text{Area of } S = \sqrt{1 + m^2 + n^2} \cdot \text{Area of } R.$$

Strengthen Your Understanding

In Problems 54–55, explain what is wrong with the statement.

54. There is only one unit vector perpendicular to two non-parallel vectors in 3-space.
55. $\vec{u} \times \vec{v} = \vec{0}$ when \vec{u} and \vec{v} are perpendicular.

In Problems 56–57, give an example of:

56. A vector \vec{u} whose cross product with $\vec{v} = \vec{i} + \vec{j}$ is parallel to \vec{k} .
57. A vector \vec{v} such that $\|\vec{u} \times \vec{v}\| = 10$, where $\vec{u} = 3\vec{i} + 4\vec{j}$.

Are the statements in Problems 58–67 true or false? Give reasons for your answer.

58. $\vec{u} \times \vec{v}$ is a vector.
59. $\vec{u} \times \vec{v}$ has direction parallel to both \vec{u} and \vec{v} .
60. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\|$.
61. $(\vec{i} \times \vec{j}) \cdot \vec{k} = \vec{i} \cdot (\vec{j} \times \vec{k})$.
62. If \vec{v} is a non-zero vector and $\vec{v} \times \vec{u} = \vec{v} \times \vec{w}$, then $\vec{u} = \vec{w}$.
63. The value of $\vec{v} \cdot (\vec{v} \times \vec{w})$ is always 0.
64. The value of $\vec{v} \times \vec{w}$ is never the same as $\vec{v} \cdot \vec{w}$.
65. The area of the triangle with two sides given by $\vec{i} + \vec{j}$ and $\vec{j} + 2\vec{k}$ is $3/2$.
66. Given a non-zero vector \vec{v} in 3-space, there is a non-zero vector \vec{w} such that $\vec{v} \times \vec{w} = \vec{0}$.
67. It is never true that $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Vectors**

Geometric definition of vector addition, subtraction and scalar multiplication, resolving into \vec{i} , \vec{j} , and \vec{k} components, magnitude of a vector, algebraic properties of addition and scalar multiplication.

- **Dot Product**

Geometric and algebraic definition, algebraic properties, using dot products to find angles and determine perpen-

dicularity, the equation of a plane with given normal vector passing through a given point, projection of a vector in a direction given by a unit vector.

- **Cross Product**

Geometric and algebraic definition, algebraic properties, cross product and volume, finding the equation of a plane through three points.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THIRTEEN

Exercises

In Exercises 1–2, is the quantity a vector or a scalar? Compute it.

- $\vec{u} \cdot \vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = \vec{k} - \vec{j}$
- $\vec{u} \times \vec{v}$, where $\vec{u} = 2\vec{i} - 3\vec{j} - 4\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$.

In Exercises 3–4, calculate the quantity.

- $(2\vec{i} - 3\vec{j} + 4\vec{k}) \cdot (2\vec{i} + 3\vec{j} + \vec{k})$
- $\vec{i} \cdot (\vec{k} \times \vec{j})$
- Resolve the vectors in Figure 13.46 into components.

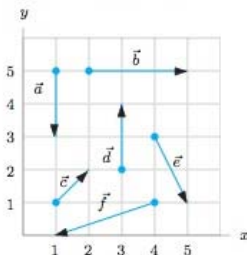


Figure 13.46

- Resolve vector \vec{v} into components if $\|\vec{v}\| = 8$ and the direction of \vec{v} is shown in Figure 13.47.

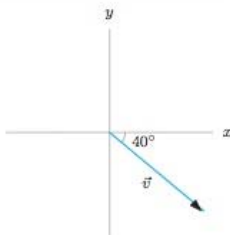


Figure 13.47

For Exercises 7–9, perform the indicated operations on the following vectors:

$$\vec{c} = \vec{i} + 6\vec{j}, \quad \vec{x} = -2\vec{i} + 9\vec{j}, \quad \vec{y} = 4\vec{i} - 7\vec{j}.$$

- $5\vec{c}$
- $\vec{c} + \vec{x} + \vec{y}$
- $\|\vec{x} - \vec{c}\|$

In Exercises 10–19, use $\vec{v} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{w} = \vec{i} - \vec{j} + 2\vec{k}$ to calculate the given quantities.

- $\vec{v} + 2\vec{w}$
- $\|\vec{v} + \vec{w}\|$
- $\vec{v} \times \vec{w}$
- $(\vec{v} \cdot \vec{w})\vec{v}$
- $(\vec{v} \times \vec{w}) \times \vec{w}$
- $3\vec{v} - \vec{w} - \vec{v}$
- $\vec{v} \cdot \vec{w}$
- $\vec{v} \times \vec{v}$
- $(\vec{v} \times \vec{w}) \cdot \vec{w}$
- $(\vec{v} \times \vec{w}) \times (\vec{v} \times \vec{w})$

In Exercises 20–21, find a normal vector to the plane.

- $2x + y - z = 23$
- $z - 5(x - 2) = 3(5 - y)$
- Find the equation of the plane through the origin which is parallel to $z = 4x - 3y + 8$.
- Let $\vec{v} = 3\vec{i} + 2\vec{j} - 2\vec{k}$ and $\vec{w} = 4\vec{i} - 3\vec{j} + \vec{k}$. Find each of the following:
 - $\vec{v} \cdot \vec{w}$
 - $\vec{v} \times \vec{w}$
 - A vector of length 5 parallel to vector \vec{v}
 - The angle between vectors \vec{v} and \vec{w}
 - The component of \vec{v} in the direction of \vec{w}
 - A vector perpendicular to vector \vec{v}
 - A vector perpendicular to both vectors \vec{v} and \vec{w}

In Exercises 24–30, find a vector with the given property.

24. Length 10, parallel to $2\vec{i} + 3\vec{j} - \vec{k}$.
25. Unit vector perpendicular to $\vec{i} + \vec{j}$ and $\vec{i} - \vec{j} - \vec{k}$.
26. Unit vector in the xy -plane perpendicular to $3\vec{i} - 2\vec{j}$.
27. Normal to $4(x-1) + 6(z+3) = 12$.
28. Perpendicular to $x - y = 1 + z$.
29. The vector obtained from $4\vec{i} + 3\vec{j}$ by rotating it 90° counterclockwise.
30. A non-zero vector perpendicular to $\vec{v} = 3\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = \vec{i} - 2\vec{j} + \vec{k}$.
31. Which of the following vectors are parallel?

$$\begin{aligned}\vec{u} &= 2\vec{i} + 4\vec{j} - 2\vec{k}, & \vec{p} &= \vec{i} + \vec{j} + \vec{k}, \\ \vec{v} &= \vec{i} - \vec{j} + 3\vec{k}, & \vec{q} &= 4\vec{i} - 4\vec{j} + 12\vec{k}, \\ \vec{w} &= -\vec{i} - 2\vec{j} + \vec{k}, & \vec{r} &= \vec{i} - \vec{j} + \vec{k}.\end{aligned}$$

In Exercises 32–37, find the parallel and perpendicular components of the force vector \vec{F} in the direction of the displacement vector \vec{d} . Then find the work W done by \vec{F} through the displacement \vec{d} .

32. $\vec{F} = 2\vec{i} + 4\vec{j}$, $\vec{d} = \vec{i} + 2\vec{j}$
33. $\vec{F} = -2\vec{i} - 4\vec{j}$, $\vec{d} = \vec{i} + 2\vec{j}$
34. $\vec{F} = 2\vec{i} + 4\vec{j}$, $\vec{d} = 2\vec{i} - 1\vec{j}$
35. $\vec{F} = 2\vec{i} + 4\vec{j}$, $\vec{d} = 3\vec{i} - 4\vec{j}$
36. $\vec{F} = 2\vec{i}$, $\vec{d} = \vec{i} + \vec{j}$
37. $\vec{F} = 5\vec{i} + 2\vec{j}$, $\vec{d} = 3\vec{j}$
38. Find the area of the triangle with vectors $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = 4\vec{i} - 2\vec{j} + \vec{k}$ as sides.

Problems

39. Figure 13.48 shows a rectangular box containing several vectors. Are the following statements true or false? Explain.

- (a) $\vec{c} = \vec{f}$ (b) $\vec{a} = \vec{d}$ (c) $\vec{a} = -\vec{b}$
 (d) $\vec{g} = \vec{f} + \vec{a}$ (e) $\vec{c} = \vec{a} - \vec{b}$ (f) $\vec{d} = \vec{g} - \vec{c}$

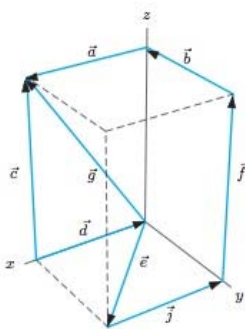


Figure 13.48

40. Shortly after takeoff, a plane is climbing northwest through still air at an airspeed of 200 km/hr, and rising at a rate of 300 m/min. Resolve its velocity vector into components. The x -axis points east, the y -axis points north, and the z -axis points up.
41. A plane is heading due east and climbing at the rate of 80 km/hr. If its airspeed is 480 km/hr and there is a wind blowing 100 km/hr to the northeast, what is the ground speed of the plane?

42. A model rocket is shot into the air at an angle with the earth of about 60° . The rocket is going fast initially but slows down as it reaches its highest point. It picks up speed again as it falls to earth.

- (a) Sketch a graph showing the path of the rocket. Draw several velocity vectors on your graph.
 (b) A second rocket has a parachute that deploys as it begins its descent. How do the velocity vectors from part (a) change for this rocket?

43. A car drives clockwise around the track in Figure 13.49, slowing down at the curves and speeding up along the straight portions. Sketch velocity vectors at the points P , Q , and R .



Figure 13.49

44. A racing car drives clockwise around the track shown in Figure 13.49 at a constant speed. At what point on the track does the car have the longest acceleration vector, and in roughly what direction is it pointing? (Recall that acceleration is the rate of change of velocity.)
45. Which pairs of the vectors $\sqrt{3}\vec{i} + \vec{j}$, $3\vec{i} + \sqrt{3}\vec{j}$, $\vec{i} - \sqrt{3}\vec{j}$ are parallel and which are perpendicular?

46. One force is pushing an object in a direction 50° south of east with a force of 25 newtons. A second force is simultaneously pushing the object in a direction 70° north of west with a force of 60 newtons. If the object is to remain stationary, give the direction and magnitude of the third force that must be applied to the object to counterbalance the first two.
47. What values of a make $\vec{v} = 2a\vec{i} - a\vec{j} + 16\vec{k}$ perpendicular to $\vec{w} = 5\vec{i} + a\vec{j} - \vec{k}$?
- In Problems 48–49, find an equation of a plane that satisfies the given conditions.
48. Perpendicular to the vector $-\vec{i} + 2\vec{j} + \vec{k}$ and passing through the point $(1, 0, 2)$.
49. Perpendicular to the vector $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passing through the point $(1, -1, 2)$.
50. Let $A = (0, 4)$, $B = (-1, -3)$, and $C = (-5, 1)$. Draw triangle ABC and find each of its interior angles.
51. Find the area of the triangle with vertices $P = (-2, 2, 0)$, $Q = (1, 3, -1)$, and $R = (-4, 2, 1)$.
52. A plane is drawn through the points $A = (2, 1, 0)$, $B = (0, 1, 3)$ and $C = (1, 0, 1)$. Find
- Two vectors lying in the plane.
 - A vector perpendicular to the plane.
 - The equation of the plane.
53. Let $P = (0, 1, 0)$, $Q = (-1, 1, 2)$, $R = (2, 1, -1)$. Find
- The area of the triangle PQR .
 - The equation for a plane that contains P , Q , and R .
54. Find the distance from the point $P = (2, -1, 3)$ to the plane $2x + 4y - z = -1$.
55. Find an equation of the plane passing through the three points $(1, 1, 1)$, $(1, 4, 5)$, $(-3, -2, 0)$. Find the distance from the origin to the plane.
56. An airport is at the point $(200, 10, 0)$ and an approaching plane is at the point $(550, 60, 4)$. Assume that the xy -plane is horizontal, with the x -axis pointing eastward and the y -axis pointing northward. Also assume that the z -axis is upward and that all distances are measured in kilometers. The plane flies due west at a constant altitude at a speed of 500 km/hr for half an hour. It then descends at 200 km/hr, heading straight for the airport.
- Find the velocity vector of the plane while it is flying at constant altitude.
 - Find the coordinates of the point at which the plane starts to descend.
 - Find a vector representing the velocity of the plane when it is descending.
57. Find the vector \vec{v} with all of the following properties:
- Magnitude 10
 - Angle of 45° with positive x -axis
 - Angle of 75° with positive y -axis
 - Positive \vec{k} -component.
58. (a) A vector \vec{v} of magnitude v makes an angle α with the positive x -axis, angle β with the positive y -axis, and angle γ with the positive z -axis. Show that
- $$\vec{v} = v \cos \alpha \vec{i} + v \cos \beta \vec{j} + v \cos \gamma \vec{k}.$$
- (b) $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are called *direction cosines*. Show that
- $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$
59. Three people are trying to hold a ferocious lion still for the veterinarian. The lion, in the center, is wearing a collar with three ropes attached to it and each person has hold of a rope. Charlie is pulling in the direction 62° west of north with a force of 175 newtons and Sam is pulling in the direction 43° east of north with a force of 200 newtons. What is the direction and magnitude of the force that must be exerted by Alice on the third rope to counterbalance Sam and Charlie?

CAS Challenge Problems

60. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$, and $\vec{c} = m\vec{a} + n\vec{b}$. Compute $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$, and explain the geometric meaning of your answers.
61. Let $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{b} = u\vec{i} + v\vec{j} + w\vec{k}$ and $\vec{c} = r\vec{i} + s\vec{j} + t\vec{k}$. Show that the parallelepiped with edges \vec{a} , \vec{b} , \vec{c} has the same volume as the parallelepiped with edges \vec{a} , \vec{b} , $2\vec{a} - \vec{b} + \vec{c}$. Explain this result geometrically.
62. Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}$, and let θ be the angle between \vec{a} and \vec{b} .
- For $\vec{c} = x\vec{i} + y\vec{j} + z\vec{k}$, write the following conditions as equations in x , y , z and solve them:

$$\vec{a} \cdot \vec{c} = 0, \quad \vec{b} \cdot \vec{c} = 0, \quad \|\vec{c}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta.$$
- [Hint: Use the dot product to find $\sin^2 \theta$.]
- Compute the cross product $\vec{a} \times \vec{b}$ and compare with your answer in part (a). What do you notice? Explain.
63. Let $A = (0, 0, 0)$, $B = (2, 0, 0)$, $C = (1, \sqrt{3}, 0)$ and $D = (1, 1/\sqrt{3}, 2\sqrt{2}/3)$.
- Show that A, B, C, D are all the same distance from each other.
 - Find the point $P = (x, y, z)$ that is equidistant from A, B, C and D by setting up and solving three equations in x, y , and z .
 - Use the dot product to find the angle APB . (In chemistry, this angle is often approximated by

109.5°. A methane molecule can be represented by four hydrogen atoms at points A, B, C and D , and a carbon atom at P .

64. Let $P = (x, y, z)$, $Q = (u, v, w)$ and $R = (r, s, t)$ be points on the plane $ax + by + cz = d$.

(a) What is the relation between $\overrightarrow{PQ} \times \overrightarrow{PR}$ and the nor-

mal vector to the plane, $a\vec{i} + b\vec{j} + c\vec{k}$?

- (b) Express $\overrightarrow{PQ} \times \overrightarrow{PR}$ in terms of $x, y, z, u, v, w, r, s, t$.
 (c) Use the equation for the plane to eliminate z, w , and t from the expression you obtained in part (b), and simplify. Does your answer agree with what you said in part (a)?

PROJECTS FOR CHAPTER THIRTEEN

1. Cross Product of Vectors in the Plane

Let $\vec{a} = a_1\vec{i} + a_2\vec{j}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j}$ be two nonparallel vectors in 2-space, as in Figure 13.50.

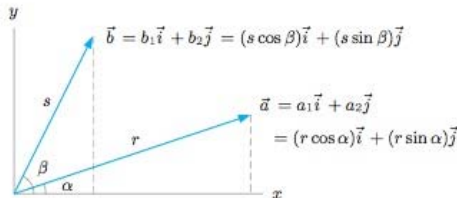


Figure 13.50

- (a) Use the identity $\sin(\beta - \alpha) = (\sin \beta \cos \alpha - \cos \beta \sin \alpha)$ to derive the formula for the area of the parallelogram formed by \vec{a} and \vec{b} :

$$\text{Area of parallelogram} = |a_1b_2 - a_2b_1|.$$

- (b) Show that $a_1b_2 - a_2b_1$ is positive when the rotation from \vec{a} to \vec{b} is counterclockwise, and negative when it is clockwise.
 (c) Use parts (a) and (b) to show that the geometric and algebraic definitions of $\vec{a} \times \vec{b}$ give the same result.

2. The Dot Product in Genetics³

We define⁴ the angle between two n -dimensional vectors, \vec{v} and \vec{w} , using the dot product:

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \frac{v_1w_1 + v_2w_2 + \cdots + v_nw_n}{\|\vec{v}\| \|\vec{w}\|}, \quad \text{provided } \|\vec{v}\|, \|\vec{w}\| \neq 0.$$

We use this idea of angle to measure how close two populations are to one another genetically. The table shows the relative frequencies of four alleles (variants of a gene) in four populations.

Allele	Eskimo	Bantu	English	Korean
A_1	0.29	0.10	0.21	0.22
A_2	0.00	0.09	0.07	0.00
B	0.03	0.12	0.06	0.21
O	0.68	0.69	0.66	0.57

³Adapted from L. L. Cavalli-Sforza and A. W. F. Edwards, "Models and Estimation Procedures," Am. J. Hum. Genet., Vol. 19 (1967), pp. 223-57.

⁴The result of Problem 70 on page 743 shows that the quantity on the right-hand side of this equation is between -1 and 1 , so this definition makes sense.

Let \vec{a}_1 be the 4-vector showing the square roots of the relative frequencies of the alleles in the Eskimo population. Let $\vec{a}_2, \vec{a}_3, \vec{a}_4$ be the corresponding vectors for the Bantu, English, and Korean populations, respectively. The genetic distance between two populations is defined as the angle between the corresponding vectors.

- Using this definition, is the English population closer genetically to the Bantus or to the Koreans? Explain.
- Is the English population closer to a half Eskimo, half Bantu population than to the Bantu population alone?
- Among all possible populations that are a mix of Eskimo and Bantu, find the mix that is closest to the English population.

3. A Warren Truss

A Warren truss is a structure for bearing a weight such as a roof or a bridge with two supports at either end of a gap. The truss in Figure 13.51 is loaded by weights at points D and E and is supported by vertical forces at points A and C . The horizontal bars in the truss are 10 ft long and the diagonal bars are 12 ft. Angles A and C are 65.38° .

Each bar exerts a force at the two joints at its ends. The two force vectors are parallel to the bar, equal in magnitude, and opposite in direction. If the bar pushes on the joints at its ends, then the bar is under compression, and if it pulls it is under tension, and the magnitude of the force is called the magnitude of the tension or compression.

Engineers need to know the magnitude of the compression or tension in each of the bars of the truss to prevent them from bending or breaking. To determine these magnitudes, we use the fact that at each joint the sum of the external forces from the weights and supports and the pushing and pulling forces exerted by the bars is zero. Find the magnitudes for all seven bars in this order:

- Joint A ; Bars AB, AE
- Joint C ; Bars BC, CD
- Joint D ; Bars BD, DE
- Joint E ; Bar BE

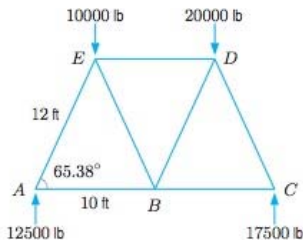


Figure 13.51