Week 7 Circuits Solutions
M&M, Spring 2015

1) Slow rate.

a) \( V(t) = V_0 \sin(2\pi f t) \). The rate of change is the (time) derivative, \( \frac{dV}{dt} = \frac{d}{dt} V_0 \sin(2\pi f t) = 2\pi f V_0 \cos(2\pi f t) \)

The largest \( \cos \) can ever get is 1, so the largest \( \frac{dV}{dt} \) will get is \( \frac{dV}{dt} = 2\pi f V_0 \), which is in Volts/sec if \( V_0 \) is in volts and \( f \) is in Hz.

b) 

![Sine Wave Graph]

The slope will be largest at the zero crossings. This can be seen in the equation above in that when \( \cos \) is a max, the \( \sin \) is a minimum... so when the sinusoidal voltage is passing zero, its derivative will be highest.

c) Well, \( \frac{dV}{dt} \) max = \( 10 \, 5 \, \text{V/s} = V_0 \cdot 2\pi f \)

and \( 10 \, \text{Vpp} = 0.5 \, \text{V amplitude} \) (i.e. \( 0.5 \, V_0 \cos \omega t \) will be \( V_0 \) volts peak-to-peak), so

\[
F \lessapprox \frac{10 \, 5 \, \text{V/s}}{2\pi \cdot 0.5 \, \text{V}} \approx 33 \, \text{kHz}, \text{ which is pretty low.}
\]
(c.ii) The highest rate of change is at the zero crossings, and for frequencies $> 33$ kHz (in this example) the sinusoid would start to get "flattened", reduced to a smaller slope than necessary. The higher the frequency above $33$ kHz, the farther the "linearization" (distortion) of the sinusoid will extend:

![flattened sinusoid diagram]

It is still curvy, since slope is smaller here.

(c.iii) By the same math as earlier, problems will now set in for $F \geq 3.3$ kHz (really low!)

$\Rightarrow$ For large signals, slew rate is often a bigger limitation than the op-amp bandwidth.

E.g., for an LF411 op-amp, the gain-bandwidth (GBW) is $3$ MHz, so for a gain of $g = 10$, the frequency bandwidth is $300$ kHz. But the LF411 has a $10 \text{ V/s}$ slew rate, so for a $\pm 15\text{V}$ signal $F \leq \frac{10^7 \text{ V/s}}{2\pi \cdot 15 \text{V}}$ or $F \leq 100$ kHz ... 3x less than the bandwidth.
Current through the resistor provides a feedback voltage directly proportional to the current.

We want \[ V_{in} = 5V \rightarrow I_{out} = 50mA. \]

Since the + and - inputs will be equal (if the op-amp can adjust its output for enough), then for \[ V_{in} = 5V, \] the voltage at "X" will also be 5V, and so the current will be

\[ I = 50mA = \frac{V}{R_1} = \frac{5V}{R_1} \quad \Rightarrow \quad R_1 = 100\, \Omega \]

If we want \[ I = 50mA \]

Note: this depends on the fact that the "-" input draws no current.
V_{out} will be 0.6 V below V_{in} when V_{in} is high, since there is no way for charge to leave the capacitor, the voltage will stay high if V_{in} drops below 0.6 V above V_{out}. So this circuit tracks the peak of V_{in}.

This circuit is even better: it will track the peak of V_{in} without any 0.6 V diode drop, since the op-amp will apply extra voltage to the diode to bring its "−" input up to V_{in}. If V_{in} drops, the op-amp will drop its output to the negative supply rail, but due to the diode no current will flow off the capacitor, and so V_{out} will remain at its previous highest value.
#3 cont'd)

For this circuit, current flowing through $R_{in}$ will be supplied by the 2nd op-amp (a "buffer") so that the charge on the capacitor $C$ will not decline and the output voltage will thus stay at peak, without any decline due to charge leaking off the capacitor through $R_{in}$.

For the previous circuit, if $V_{in} < V_{out}$, then $V_{out}$ would exponentially decay to the new $V_{in}$ with time constant $T = R_{in} \cdot C$.

#4)

Since the "-" input must be at ground (since the "+" input is grounded), and the inputs draw no current (Golden Rules #1 and #2), the currents at point "X" must add up to zero:

$$I_1 + I_2 + I_3 = \frac{V_1}{R} + \frac{V_2}{R} + \frac{V_{out}}{R} = 0$$

So $V_1 + V_2 = -V_{out}$, an "adder" with inverted output.
#5) a) \[ V_{in1} \rightarrow \text{point } A \rightarrow V_{out} \]

\[ \text{From the Golden Rules, we know the Voltage at point } A, \quad V_A = 0 \text{ since the } + \text{ input is grounded.} \]

\[ \text{We also know that the } - \text{ input draws no current (the other golden rule), so} \quad I_1 = I_2, \quad \text{where} \]

\[ I_1 = \frac{(V_{in1} - V_A)}{R_1}, \quad I_2 = \frac{V_A - V_{out}}{R_2} \]  

(1)

Since \( V_A = 0 \) here, this becomes

\[ \frac{V_{in1}}{R_1} = -\frac{V_{out}}{R_2} \]  

(2)

or

\[ G = \frac{V_{out}}{V_{in1}} = -\frac{R_2}{R_1} \]  

(3) a standard inverting amplifier

b)

\[ \text{Here } V_A = V_B = V_{in2}, \text{ based on the Golden Rules} \]

Thus we know

\[ I_1 = \frac{0 - V_A}{R_1} = -\frac{V_{in2}}{R_1} \]  

(4)
and \[ I_z = \frac{V_A - V_{out}}{R_z} \] (5)

But by the golden rules the input draws no current, so \[ I_1 = I_z \] (6)

and \[ -\frac{V_{in} + V_{in}}{R_1} = \frac{V_{in} - V_{out}}{R_2} \] (7) since \( V_A = V_{in} \)

or \[ \frac{V_{out}}{R_2} = V_{in} + \left[ \frac{1}{R_2} + \frac{1}{R_1} \right] \] (8)

or \[ G = \frac{V_{out}}{V_{in} + V_{out}} = 1 + \frac{R_2}{R_1} \] (9) ... standard for a noninverting amplifier !)

\[ \Rightarrow \text{you can now see that if you just flip the circuit over,} \]

\[ V_{in} \]

\[ + \]

\[ \frac{1}{2} R_2 \]

\[ \frac{1}{2} R_1 \]

\[ \text{point A} \]

\[ V_{out} \]

\[ \text{Fig 2} \]

it is the same as Fig 1 ...

note that looked at this way, we know \[ V_A = V_{out} + \frac{R_{in}}{R_{1} + R_{2}} \] (10) since it's a voltage divider

and since \( V_A = V_{in} \) (from the golden rules)
we get \[ \frac{V_{\text{out}}}{V_{i\text{.2}}} = \frac{R_1}{R_1+R_2} \]

\[ G = \frac{V_{\text{out}}}{V_{i\text{.2}}} = 1 + \frac{R_2}{R_1} \]

The same result as before...

so there are multiple ways to solve this.

Still, useful to look and see if by rearranging a circuit you can get it into a form for which you already know the answer.

![Circuit Diagram](image)

There are (at least) two ways to do this...

Easy way: If you ground \( V_{i\text{.1}} \), then from (b) we know

\[ V_{\text{out}} = (1 + \frac{R_2}{R_1}) V_B \]

Furthermore, \( V_{i\text{.2}} \) is just a voltage divider...

So

\[ V_B = \frac{R_2}{R_1+R_2} V_{i\text{.2}} \]  \[ (1) \]
and so \( V_{\text{out}} = \left(1 + \frac{R_2}{R_1}\right) \left(\frac{R_2}{R_1 + R_2}\right) V_{i1} - V_{i2} \) (12)

or
\[
V_{\text{out}} = \frac{R_2}{R_1} V_{i1} - \frac{R_2}{R_1} V_{i2} \quad \text{(using (13) and (14))}
\]

or
\[
V_{\text{out}} = \frac{R_2}{R_1} \left[ V_{i1} - V_{i2} \right] \quad \text{(16)}
\]

\( \Rightarrow \) i.e., the circuit is a "differential amplifier"; amplifying the difference between \( V_i1 \) and \( V_i2 \).

Harder Way:

It may bother you to use the principle of superposition on a circuit that has an op-amp made of transistors, since they are not linear devices and superposition depends on the absolute linearity of a system (of circuits or equations). So why is this OK?
#5 cont'd

The short answer is that the feedback has linearized the system. Remember that

\[ V_{in} - B V_{out} \]

\[ A (V_{in} - B V_{out}) = V_{out} \]

So

\[ A (V_{in} - B V_{out}) = V_{out} \quad (17) \]

\( A \) may vary a lot with voltage or frequency—i.e., be nonlinear.

Rearranging this,

\[ A V_{in} = V_{out} + A B V_{out} \]

or

\[ G = \frac{V_{out}}{V_{in}} = \frac{A}{1 + A B} \quad (18) \]

\[ G = \frac{1}{B} \quad \text{if} \ A \to \infty \quad (19) \]

(really, if \( AB \gg 1 \))

So as long as \( A \) is really big (usually \( A \), the "open loop gain" is \( > 100,000 \) or so) then all the nonlinearity of \( A \) gets lost and \( G \approx \frac{1}{B} \) is linear (if \( B \) is... and you usually make \( B \) out of linear devices like..."
resistors, etc. Since the overall behavior of the circuit is now linear, you can get away with superposition.

Still, if it bothers you, note that in Fig 3 (p. 8)

$$V_b = V_{in2} \frac{R_2}{R_1+R_2} \quad (20)$$

and go back to p. 6, Eqn (1), where we had

$$I_1 = I_2 = \frac{V_{in1}-V_A}{R_1} = \frac{V_A-V_{out}}{R_2} \quad (21) \quad \text{(also (1))}$$

since now $V_A = V_B$ (Golden rule: op-amp will shift $V_{out}$ to get inputs to match, if possible), (20) and (21) give

$$\frac{V_{in1}-V_{in2}}{R_1} = V_{in2} \frac{R_2}{R_1+R_2} - \frac{V_{out}}{R_2} \quad (22)$$

or

$$\frac{V_{out}}{R_2} = V_{in2} \left[ \frac{1}{R_1+R_2} + \frac{R_2}{R_1(R_1+R_2)} \right] - \frac{V_{in1}}{R_1} \quad (23)$$

$$= V_{in2} \left[ \frac{1}{(R_1+R_2)} \left( 1 + \frac{R_2}{R_1} \right) \right] - \frac{V_{in1}}{R_1}$$

$$= \frac{V_{in2}}{R_1} - \frac{V_{in1}}{R_1}$$

so

$$V_{out} = \frac{R_2}{R_1} \left[ V_{in2} - V_{in1} \right] \quad (24) \quad \text{as before.}$$
From Eqs. (13) - (16), p. 9,

\[ V_{\text{out}} = -\frac{R_2}{R_1} V_{\text{in}1} + \frac{R_{41}}{R_3+R_4} \cdot \frac{R_1+R_2}{R_1} V_{\text{in}2} \quad (25) \]

since we are looking for the effects on the "common mode" input, i.e., on the component \( V_{\text{in}1} = V_{\text{in}2} = V_{\text{cm}} \),

we can rewrite this as

\[ V_{\text{out}} = V_{\text{cm}} \left[ \frac{R_4}{R_3+R_4} \cdot \frac{R_1+R_2}{R_1} - \frac{R_2}{R_1} \cdot \frac{R_3+R_4}{R_3+R_4} \right] \quad (26) \]

or

\[ V_{\text{out}} = V_{\text{cm}} \left[ \frac{R_1 R_4 - R_3 R_2}{R_1 (R_3+R_4)} \right] \quad (27) \]

An approximate "worst case" way to deal with this is to lump the errors as follows:

\[
\begin{align*}
R_1 &= R (1 + \delta_1) \\
R_2 &= R (1 + \delta_2) \\
R_3 &= R (1 + \delta_3) \\
R_4 &= R (1 + \delta_4)
\end{align*}
\]

where \( \delta_n = \) fractional error for that resistor.

For 0.01%, \( \delta = \pm 1 \times 10^{-4} \).
Since \( \delta_n \ll 1 \), small errors in the denominator won't matter, and we can ignore them in a first approximation (because \( \frac{1}{1+\delta} \approx 1-\delta \approx 1 \)). Thus inserting relations (28) into (27) gives

\[
V_{out} \approx V_{cm} \left[ (1+\delta_1)(1+\delta_4) - (1+\delta_3)(1+\delta_2) \right] \quad (29)
\]

\[
\approx V_{cm} \left[ (1+\delta_1+\delta_4) - (1+\delta_3+\delta_2) \right]
\]

where I dropped terms in \( \delta^2 \) since they are \( \ll \delta \) (since \( \delta \ll 1 \) already). So

\[
V_{out} \approx V_{cm} \left[ \delta_1+\delta_4 - \delta_3 - \delta_2 \right] \quad (30)
\]

The absolute worst case would be if all these random \( \delta = \pm 1e^{-4} \) errors added up to a maximum, i.e.

\[
V_{out,\text{worst case}} \approx V_{cm} \cdot 4\delta \approx 4e^{-4} \cdot V_{cm} \quad (31)
\]

In dB,

\[
20 \log \left( \frac{V_{out}}{V_{cm}} \right) = 20 \log \left( 4e^{-4} \right)
\]

\[
= 20 \log \left( 4 \cdot 10^{-4} \right) = 20 \left( 4 + \log 10^4 \right)
\]

\[
= 20 \left( 4 + \log (2.2) \right) = 20 \left( 4.6 \right) = 92 \text{ dB}
\]

So the CMRR would be better than 92 dB.

There are of course, more sophisticated (and complicated) statistical ways of doing this.