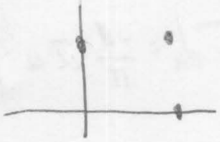
Calculus Week 19 solutions - odd

8.5: 7, 11, 13, 21, 25, 27, 29   8.6: 5, 7, 13, 21

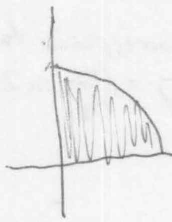
7) total mass = 3 (three unit masses) $M_x = \sum m_n x_n = 1\cdot 1 + 1\cdot 1 + 1\cdot 0 = 2$

$\bar{x} = \frac{\text{moment}}{\text{mass}} = \frac{2}{3}$



$M_y = \sum m_n y_n = \frac{2}{3}$

11)



$y = \sqrt{1-x^2}$

$M = \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int_0^{\pi/2} \cos^2\theta\,d\theta$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

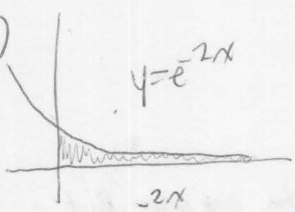
$= \int_0^{\pi/2} \frac{\cos 2\theta + 1}{2}\,d\theta = \frac{\sin 2\theta}{4}\Big]_0^{\pi/2} + \frac{\theta}{2}\Big]_0^{\pi/2} = \frac{\pi}{4}$

$\bar{x} = \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 -\frac{\sqrt{u}}{2}\,du = \frac{-u^{3/2}}{3}\Big]_1^0 = \frac{1}{3}\cdot\frac{4}{\pi} = \frac{4}{3\pi}$

$u = 1-x^2$
$du = -2x\,dx$

$\bar{y} = \frac{4}{3\pi}$ by symmetry

13)



$M = \int_0^\infty e^{-2x}\,dx = \frac{-e^{-2x}}{2}\Big]_0^\infty = \frac{1}{2}$

$\bar{x} = \frac{\int_0^\infty x e^{-2x}\,dx}{\frac{1}{2}} = 2\int_0^\infty x e^{-2x}\,dx = 2\,\frac{xe^{-2x}}{-2}\Big]_0^\infty + \int_0^\infty \frac{e^{-2x}}{2}\,dx = -xe^{-2x} + \frac{e^{-2x}}{-4}\Big]_0^\infty$

$= \frac{1}{4}$

$u = x \quad dv = e^{-2x}\,dx$
$du = dx \quad v = \frac{e^{-2x}}{-2}$

$y = e^{-2x}$
$\Rightarrow \ln y = -2x$
$\Rightarrow x = \frac{-\ln y}{2}$

So $\bar{y} = 2\int_0^1 y\left(\frac{-\ln y}{2}\right)dy = \frac{-2}{2}\int_0^1 y\ln y\,dy = \frac{\ln y\, y^2}{2}\Big]_0^1 + \int_0^1 \frac{y}{2}\,dy = \frac{\ln y\cdot y^2}{-2} + \frac{y^2}{4}\Big]_0^1 = +\frac{1}{4}$

let $u = \ln y \quad dv = y\,dy$
$du = \frac{1}{y}dy \quad v = \frac{y^2}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{1}{4}, \frac{1}{4}\right)$

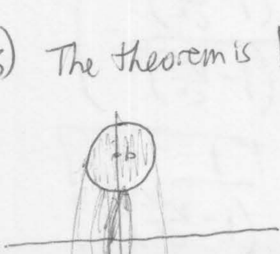
21)

$I = \int (x-t)^2 \rho(x)\,dx = \int x^2\rho(x) - 2xt\rho(x) + t^2\rho(x)\,dx$ so $\frac{dI}{dt} = \frac{d}{dt}\int x^2\rho(x) - 2xt\rho(x) + t^2\rho(x)\,dx$

$= -\int 2x\rho(x)\,dx + 2t\int \rho(x)\,dx$

If $t = \bar{x} = \frac{\int x\rho(x)\,dx}{\int \rho(x)\,dx}$ then $\frac{dI}{dt} = -\int 2x\rho(x)\,dx + \frac{2\int x\rho(x)\,dx}{\int \rho(x)\,dx}\cdot\int \rho(x)\,dx = -\int 2x\rho(x)\,dx + \int 2x\rho(x)\,dx = 0$

25) The theorem is $V = \int 2\pi y\,(\text{strip width})\,dy = 2\pi M_x = 2\pi\bar{y}M$



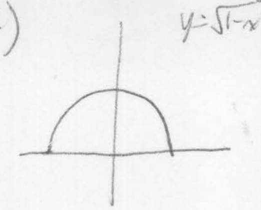
In this case, $M = \text{area} = \pi a^2$ and $\bar{y} = b$

so $V = 2\pi^2 b a^2$

torus (donut)

8.5 #27

27)



$$M = \rho \cdot L = 1 \cdot \pi = \pi$$

$M_x = 0$ by symmetry

$$M_y = 2\int x\,ds = 2\int_0^1 x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\int_0^1 x\sqrt{1+\left(\frac{-2x}{2\sqrt{1-x^2}}\right)^2}\,dx = 2\int_0^1 x\sqrt{1+\frac{x^2}{\sqrt{1-x^2}^{\,2}}}\,dx$$

$$= 2\int_0^1 x\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 2\int_0^1 x\sqrt{\frac{1}{1-x^2}}\,dx = -2\int_1^0 -\frac{1}{2}\sqrt{\frac{1}{u}}\,du = -2u^{\frac{1}{2}}\Big]_1^0 = 2$$

$u = 1-x^2$
$du = -2x\,dx$

29) We check our answer to #27 by using the theorem from Problem 28: $A = 2\pi\bar{y}L = 4\pi \Rightarrow \bar{y}L = 2 \Rightarrow \bar{y}\cdot\pi = 2 \Rightarrow \bar{y} = \frac{2}{\pi}$ ✓ (Corresponds to $\bar{y} = \frac{M_y}{M}$ in 27)

8.6: 5, 7, 13, 21

5) (a) $W = F\cdot s = \frac{120x}{12}$ foot-pounds $= 10x$ foot-pounds

(b) Also $\frac{120x}{12}$ foot-pounds $= 10x$ foot-pounds

7) $$W = \int F\,ds = \int_0^{100} 5x\,dx = \frac{5x^2}{2}\Big]_0^{100} = \frac{50000}{2} = 25{,}000 \text{ foot-pounds.}$$

With the 200-lb weight we add 20000 to get 45,000 foot-pounds

13) $W = Fs = \int kx\,dx \Rightarrow 20 = \frac{kx^2}{2}\Big]_0^2 = 2k \Rightarrow k = 10$. Therefore, to stretch one more foot we do $W = \int_0^3 kx\,dx = \frac{10x^2}{2}\Big]_0^3 = 5x^2\Big]_0^3 = 45$ foot pounds, or 25 more foot pounds

21) $m = m_0 \big/ \sqrt{1-\frac{v^2}{c^2}}$ Einstein: $F = \frac{d(mv)}{dt} = \frac{d(mv)}{dv}\cdot\frac{dv}{dt} = \frac{d}{dv}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right)\cdot a$

$$= m_0 a\left(\frac{-v\left(\pm\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(\frac{-2v}{c^2}\right)\right) + \sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}\right)$$ (quotient rule)

$$= m_0 a\left(\frac{+\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}\right)\cdot\left(\frac{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}\right)$$

$$= m_0 a\left(\frac{+\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right) = m_0 a\,\boxed{\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}}$$

conversion factor