Calculus Week 16 solutions - odd

7.1 3, 5, 7, 9, 13, 19, 23, 31, 55   7.2 3, 15, 39   7.3 3, 5, 13, 21, 23, 53

3) $\int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} + (-e^{-x}) + C$

$u = x \quad dv = e^{-x}$
$du = dx \quad v = -e^{-x}$

5) $\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx = x^2 \sin x - 2\overbrace{\int x \sin x\,dx}^{\text{solved previously}}$

$= x^2 \sin x + 2x\cos x + 2\sin x + C$

$u = x^2 \quad dv = \cos x\,dx$
$du = 2x\,dx \quad v = \sin x$

7) $\int \ln(2x+1)\,dx = x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{u-1}{2u}\,du$

$= x\ln(2x+1) - \int \frac{1}{2} - \frac{1}{2u}\,du$

$= x\ln(2x+1) - \frac{u}{2} - \frac{1}{2}\ln u + C$

$= x\ln(2x+1) - \frac{2x+1}{2} - \frac{1}{2}\ln(2x+1) + C$

$u = \ln(2x+1) \quad dv = dx$
$du = \frac{2}{2x+1}\,dx \quad v = x$

Subst: $u = 2x+1$
$du = 2\,dx$

9) $\int e^x \sin x\,dx = -e^x \cos x + \int \cos x\, e^x\,dx = -e^x\cos x + e^x \sin x - \underbrace{\int \sin x\, e^x\,dx}_{\text{add to left hand side}}$

$u = e^x \quad dv = \sin x\,dx$
$du = e^x \quad v = -\cos x$

$u = e^x \quad dv = \cos x\,dx$
$du = e^x \quad v = \sin x$

$\Rightarrow 2\int \sin x\, e^x\,dx = -e^x\cos x + e^x \sin x \quad$ so $\int \sin x\, e^x\,dx = \frac{-e^x\cos x + e^x \sin x}{2} + C$

13) $\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \frac{1}{x}\, x\cos(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

$u = \sin(\ln x) \quad dv = dx$
$du = \frac{1}{x}\cos(\ln x) \quad v = x$

$u = \cos(\ln x) \quad dv = dx$
$du = \frac{1}{x}(-\sin(\ln x)) \quad v = x$

$= x\sin(\ln x) - x\cos(\ln x) - \int \frac{1}{x}\, x\sin(\ln x)\,dx$

$\Rightarrow 2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C$

19) $\int x \tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2(1+x^2)}\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2 \overbrace{+1-1}^{\text{add } 0}}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{1+x^2}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$

$u = \tan^{-1}x \quad dv = x\,dx$
$du = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}$

#23 $\int x^3 e^x dx = x^3 e^x - \int 3x^2 e^x = x^3e^x - 3\left(x^2e^x + \int 2xe^x dx\right) = x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x dx$

$u = x^3 \quad dv = e^x dx \qquad u = x^2 \quad dv = e^x dx \qquad u = x \quad dv = e^x dx$

$du = 3x^2 dx \quad v = e^x \qquad du = 2x\,dx \quad v = e^x \qquad du = dx \quad v = e^x$

$= (x^3 - 3x^2 + 6x - 6)e^x + C$

#31 $\int_0^\pi x\cos x\, dx = x\sin x\Big]_0^\pi - \int_0^\pi \sin x\, dx = 0 + \cos x\Big]_0^\pi = \cos\pi - \cos 0 = -1-1 = -2$

$u = x \quad dv = \cos x\, dx$

$du = dx \quad v = \sin x$

#55

$\int uv'dx = uv - Vu' + \int V u'' dx$

Example $\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + \int -\sin x \cdot 2\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

$u = x^2 \quad v' = \sin x \qquad$ so $u'' = 2$

$du = 2x\,dx \quad v = -\cos x \qquad V = -\sin x$

$u' = 2x$

Check by differentiating:

$-2x\cos x + x^2\sin x + 2\sin x + 2x\cos x - 2\sin x$

$= x^2 \sin x$ ✓

7.2 #3, 15, 39

#3 $\int \sin x \cos x\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$

subst. $u = \sin x$

$du = \cos x\, dx$

#15 $\int \sin^2 x + \int \cos^2 x$ (we know this should be $\int 1\, dx = x + C$)

Using double angle formulas: $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$ so $\cos^2 x = \frac{\cos 2x + 1}{2}$

or $= 1 - 2\sin^2 x \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$

So $\int \sin^2 x\, dx + \int \cos^2 x\, dx = \int \frac{1 - \cos 2x}{2} dx + \int \frac{\cos 2x + 1}{2} dx = \int \frac{1}{2} - \frac{1}{2}\int \cos 2x + \frac{1}{2}\int \cos 2x + \int \frac{1}{2}$

$= \frac{x}{2} - \frac{\sin 2x}{4} + \frac{\sin 2x}{4} + \frac{x}{2} + C$

$= x + C$ ✓

#39 We know $\int_0^{2\pi} \sin p_0 x \cos q_0 x\, dx = 0$. Let $u = \frac{x}{2}$, $du = \frac{dx}{2}$. Then $\int_0^\pi (\sin p_0 \cdot 2u \cos q_0 \cdot 2u)\left(\frac{1}{2}\right) dx = 0$

This means $\int_0^\pi \sin px \cos qx\, dx = 0$ whenever p and q are both even

7.3: 3, 5, 13, 21, 25, 53

3) $\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = \int 2\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\,d\theta$

Let $x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

$= 4\int \cos^2\theta\,d\theta = 4\int \frac{\cos(2\theta)+1}{2}\,d\theta$

$= 4\int \frac{\cos 2\theta}{2} + 2\int d\theta = \sin 2\theta + 2\theta + C$

$= x\sqrt{1-x^2} + \sin^{-1}\frac{x}{2} + C$

5) $\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^2\theta\,d\theta = \int \frac{1}{2}(1-\cos 2\theta)\,d\theta$

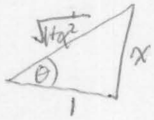
$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{4}\right) = \frac{1}{2}\left(\sin^{-1}x - 2x\sqrt{1-x^2}\right) + C$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

13) $\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{1}{(1+\tan^2\theta)^{3/2}}\cdot\sec^2\theta\,d\theta = \int \frac{1}{\sec^3\theta}\cdot\sec^2\theta\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C$

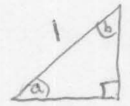
$= \frac{x}{\sqrt{1+x^2}} + C$

$x = \tan\theta$
$dx = \sec^2\theta\,d\theta$



21) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\cos^2\theta}}(-\sin\theta\,d\theta) = \int \frac{-\sin\theta}{\sqrt{\sin^2\theta}}\,d\theta = -\int \frac{\sin\theta}{\sin\theta}\,d\theta = -\int d\theta = -\theta = -\cos^{-1}x + C$

$x = \cos\theta \quad dx = -\sin\theta\,d\theta$



$a = \cos^{-1}x \quad a = 90° - b$
$b = \sin^{-1}x$ so $-a$ and b differ by a constant

25) $\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta\,d\theta = a^2 \underbrace{\left(\frac{\sin 2\theta}{2} + \frac{\theta}{2}\right)}_{\text{solved previously}}\Big]_{-\pi/2}^{\pi/2} = a^2\cdot\frac{\pi}{2} = \frac{1}{2}\pi a^2$ (area of a semicircle of radius a)

$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$

53) $\int_0^{2\pi} \cos^2\theta\,d\theta \neq \int_{\sin 0}^{\sin 2\pi} \sqrt{1-x^2}\,dx$ $\boxed{\text{since } \cos\theta = \pm\sqrt{1-x^2} \text{ depending on } \theta.}$ ← answer

$x = \sin\theta$ so $\cos\theta = \sqrt{1-x^2}$ when $\cos\theta > 0$, so when $0 < \theta < \pi/2$ or $3\pi/2 < \theta < 2\pi$
$dx = \cos\theta\,d\theta$ and $\cos\theta = -\sqrt{1-x^2}$ when $\cos\theta < 0$, so $\pi/2$ to $3\pi/2$

so, instead: $\int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{\sin\pi/2} +\sqrt{1-x^2}\,dx + \int_{\sin\pi/2}^{\sin 3\pi/2} -\sqrt{1-x^2}\,dx + \int_{\sin 3\pi/2}^{\sin 2\pi} \sqrt{1-x^2}\,dx$

$= \int_0^1 \sqrt{1-x^2}\,dx - \int_1^{-1} \sqrt{1-x^2} + \int_{-1}^0 \sqrt{1-x^2} = \pi$ (area circle)

addl work to solve the integral

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