5.7 The Fundamental Theorem and Its Applications (page 219)

The area $f(x) = \int_a^x v(t)\,dt$ is a function of $\mathbf{x}$. By Part **1** of the Fundamental Theorem, its derivative is $\mathbf{v(x)}$. In the proof, a small change Δx produces the area of a thin **rectangle.** This area Δf is approximately $\mathbf{\Delta x}$ times $\mathbf{v(x)}$. So the derivative of $\int_a^x t^2\,dt$ is $\mathbf{x^2}$.

The integral $\int_x^b t^2\,dt$ has derivative $\mathbf{-x^2}$. The minus sign is because **x is the lower limit.** When both limits $a(x)$ and $b(x)$ depend on x, the formula for df/dx becomes $\mathbf{v(b(x))\frac{db}{dx}}$ minus $\mathbf{v(a(x))\frac{da}{dx}}$. In the example $\int_2^{3x} t\,dt$, the derivative is $\mathbf{9x}$.

By Part **2** of the Fundamental Theorem, the integral of df/dx is $\mathbf{f(x) + C}$. In the special case when $df/dx = 0$, this says that **the integral is constant.** From this special case we conclude: If $dA/dx = dB/dx$ then $A(x) = \mathbf{B(x) + C}$. If an antiderivative of $1/x$ is $\ln x$ (whatever that is), then automatically $\int_a^b dx/x = \mathbf{\ln b - \ln a}$.

The square $0 \le x \le s, 0 \le y \le s$ has area $A = \mathbf{s^2}$. If s is increased by Δs, the extra area has the shape of **an L.** That area ΔA is approximately $\mathbf{2s\ \Delta s}$. So $dA/ds = \mathbf{2s}$.

1 $\cos^2 x$ **3** 0 **5** $(x^2)^3(2x) = 2x^7$ **7** $v(x+1) - v(x)$ **9** $\frac{\sin^2 x}{x} - \frac{1}{x^2}\int_0^x \sin^2 t\,dt$
11 $\int_0^x v(u)du$ **13** 0 **15** 0 **17** $u(x)v(x)$ **19** $\sin^{-1}(\sin x)\cos x = x\cos x$
21 F; F; F; T **23** Taking derivatives $v(x) = (x\cos x)' = \cos x - x\sin x$
25 Taking derivatives $-v(-x)(-1) = v(x)$ so v is even **27** F; T; T; F
29 $\int_1^x v(x)dx = \int_0^x v(x)dx - \int_0^1 v(x)dx = \frac{1}{x+2} - \frac{1}{1+2}$
31 $V = s^3; A = 3s^2$; half of hollow cube; $\Delta V \approx 3s^2dS; 3s^2$ (which is A)
33 $dH/dr = 2\pi^2r^3$ **35** Wedge has length $r \approx$ height of triangle; $\int_0^{\pi/2} \frac{1}{2}r^2d\theta = \frac{\pi r^2}{4}$
37 $r = \frac{1}{\cos\theta}; \frac{d\theta}{2\cos^2\theta}; \int_0^{\pi/4} \frac{d\theta}{2\cos^2\theta} = \frac{\tan\theta}{2}\Big]_0^{\pi/4} = \frac{1}{2}$
39 $x = y^2; \int_0^2 y^2dy = \frac{y^3}{3}\Big]_0^2 = \frac{8}{3}$; vertical strips have length $2 - \sqrt{x}$
41 Length $\sqrt{2}a$; width $\frac{da}{\sqrt{2}}; \int_0^1 a\,da = \frac{1}{2}$ **43** The differences of the sums $f_j = v_1+v_2+\cdots+v_j$ are $f_j - f_{j-1} = v_j$
45 No, $\int_0^x a(t)dt = \frac{df}{dx}(x) - \frac{df}{dx}(0)$ and $\int_0^1(\int_0^x a(t)dt)dx = f(1) - f(0) - \frac{df}{dx}(0)$

2 $\frac{d}{dx}\int_x^1 \cos 3t\,dt = -\cos \mathbf{3x}$. **4** $\frac{d}{dx}\int_0^2 x^n dt = \frac{d}{dx}2x^n = \mathbf{2nx^{n-1}}$.
6 $\frac{d}{dx}\int_{-x}^{x/2} v(u)du = \frac{1}{2}v(\frac{x}{2}) - (-1)v(-x) = \mathbf{\frac{1}{2}v(\frac{x}{2}) + v(-x)}$
8 $\frac{d}{dx}(\frac{1}{x}\int_0^x v(t)dt)$ by the product rule is $\mathbf{\frac{1}{x}v(x) - \frac{1}{x^2}\int_0^x v(t)dt}$ which is $\frac{1}{x^2}\int_0^x(v(x) - v(t))dt$.
10 $\frac{d}{dx}(\frac{1}{2}\int_x^{x+2} x^3dx) = \mathbf{\frac{1}{2}(x+2)^3 - \frac{1}{2}x^3}$ **12** $\frac{d}{dx}\int_0^x(\frac{df}{dx})^2dx = \mathbf{(\frac{df}{dx})^2(x)}$ **14** $\frac{d}{dx}\int_0^x v(-t)dt = \mathbf{v(-x)}$
16 $\frac{d}{dx}\int_{-x}^x \sin t\,dt = \sin x - (-1)\sin(-x) = \mathbf{0}$. (The integral is zero because $\sin t$ is odd)
18 $\frac{d}{dx}\int_{a(x)}^{b(x)} 5dt = 5\frac{db}{dx} - 5\frac{da}{dx}$. **20** $\frac{d}{dx}(\int_0^{f(x)} \frac{df}{dt}dt) = \frac{d}{dx}f(f(x)) = \mathbf{f'(f(x))f'(x)}$.
22 $F(\pi + \Delta x) - F(\pi)$ is the strip of width $2\Delta x$ beyond $x = 2\pi$ on the sine graph minus the strip of width Δx beyond $x = \pi$ (compare Figure 5.15b). $F(\Delta x) - F(0)$ is the strip from Δx to $2\Delta x$.
24 If $\frac{df}{dx} = 2x$ then the derivative of $f(x) - x^2$ is zero. So $f(x) - x^2$ is a constant C (this was the point of equation (7)).
26 $\int_{2x}^{3x} \frac{dt}{t} = \int_{u=2}^{3} \frac{x\,du}{xu} = \mathbf{\int_2^3 \frac{du}{u}}$ (which is a number – not dependent on x). **28** $\int_1^x v(x)dx = x^n\Big]_1^x = \mathbf{x^n - 1}$.
30 When the side s is increased, only *two* strips are added to the square (on the right side and top). So $dA = 2s\ ds$

which agrees with $A = s^2$.

32 The 4-dimensional cube has volume $H = \mathbf{s^4}$. The face with $x = s$ is a **3-dimensional cube**. Its volume is $V = s^3$. Four faces have volume $\mathbf{4s^3}$. Increase by Δs gives $\Delta H = \mathbf{(s + \Delta s)^4} - s^4$. So $dH/ds = \mathbf{4s^3}$.

34 $\int x\, dy = \int_0^1 \sqrt{y}\, dy = \frac{2}{3}y^{3/2}]_0^1 = \frac{2}{3}$.

36 A is the area under $y = \sqrt{r^2 - x^2}$ (quarter of a circle). Then $\int_{x=0}^{r} \sqrt{r^2 - x^2}dx = \int_{\theta=0}^{\pi/2}(r\cos\theta)(r\cos\theta\, d\theta) = \mathbf{\frac{\pi}{4}r^2}$ because the average value of $\cos^2\theta$ is $\frac{1}{2}$. (Its integral is $\frac{1}{2}(\theta + \sin\theta\cos\theta)]_0^{\pi/2} = \frac{\pi}{4}$.)

38 The triangle ends at the line $x + y = 1$ or $r\cos\theta + r\sin\theta = 1$. The area is $\frac{1}{2}$, by geometry. So the area integral $\int_{\theta=0}^{\pi/2} \frac{1}{2}r^2 d\theta = \frac{1}{2}$: Substitute $r = \frac{1}{\cos\theta + \sin\theta}$.

40 Rings have area $2\pi r\, dr$, and $\int_2^3 2\pi r\, dr = \pi r^2]_2^3 = 5\pi$. Strips are difficult because they go in and out of the ring (see Figure 14.5b on page 528).

42 The strip around the ellipse does not have constant width dr. The width is dr in the x direction and 2 dr in the y direction.

44 The sum to $j = n$ of the differences $f_j - f_{j-1}$ is $f_n + C$ (and the constant is $C = -f_0$). This sum telescopes: $(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) \cdots$

46 At $t = 1$ the area is under the parabola $y = -x^2 + 1$. The line along the base has length $\frac{dA}{dt}$, because an increase Δt raises the mountain by Δt and adds a strip along the base. These strips have increasing length so $\frac{d}{dt}(\frac{dA}{dt}) > 0$.

5.8 Numerical Integration (page 226)

To integrate $y(x)$, divide $[a, b]$ into n pieces of length $\Delta x = \mathbf{(b - a)/n}$. R_n and L_n place a **rectangle** over each piece, using the height at the right or **left** endpoint: $R_n = \Delta x(y_1 + \cdots + y_n)$ and $L_n = \mathbf{\Delta x(y_0 + \cdots + y_{n-1})}$. These are **first**-order methods, because they are incorrect for $y = \mathbf{x}$. The total error on $[0, 1]$ is approximately $\mathbf{\frac{\Delta x}{2}(y(1) - y(0))}$. For $y = \cos\pi x$ this leading term is $\mathbf{-\Delta x}$. For $y = \cos 2\pi x$ the error is very small because $[0, 1]$ is a complete **period**.

A much better method is $T_n = \frac{1}{2}R_n + \mathbf{\frac{1}{2}L_n} = \Delta x[\frac{1}{2}y_0 + \mathbf{1}y_1 + \cdots + \mathbf{\frac{1}{2}}y_n]$. This **trapezoidal** rule is **second**-order because the error for $y = x$ is **zero**. The error for $y = x^2$ from a to b is $\mathbf{\frac{1}{6}(\Delta x)^2(b - a)}$. The **midpoint** rule is twice as accurate, using $M_n = \Delta x[\mathbf{y_{\frac{1}{2}}} + \cdots + \mathbf{y_{n-\frac{1}{2}}}]$.

Simpson's method is $S_n = \frac{2}{3}M_n + \mathbf{\frac{1}{3}T_n}$. It is **fourth**-order, because the powers $\mathbf{1, x, x^2, x^3}$ are integrated correctly. The coefficients of $y_0, y_{1/2}, y_1$ are $\mathbf{\frac{1}{6}, \frac{4}{6}, \frac{1}{6}}$ times Δx. Over three intervals the weights are $\Delta x/6$ times $1 - 4 - \mathbf{2} - \mathbf{4} - \mathbf{2} - \mathbf{4} - \mathbf{1}$. Gauss uses **two** points in each interval, separated by $\Delta x/\sqrt{3}$. For a method of order p the error is nearly proportional to $\mathbf{(\Delta x)^p}$.

1 $\frac{1}{2}\Delta x(v_0 - v_n)$ **3** $1, .5625, .3025; 0, .0625, .2025$ **5** $L_8 \approx .1427, T_8 \approx .2052, S_8 \approx .2000$
7 $p = 2 :$ for $y = x^2, \frac{1}{4} \cdot 0^2 + \frac{1}{2} \cdot (\frac{1}{2})^2 + \frac{1}{4} \cdot 1^2 \neq \frac{1}{3}$ **9** For $y = x^2$, error $\frac{1}{6}(\Delta x)^2$ from $\frac{1}{2} - \frac{1}{3}, y_1' = 2\Delta x$
13 8 intervals give $\frac{(\Delta x)^2}{12}[-\frac{1}{b^2} + \frac{1}{a^2}] = \frac{1}{1024} < .001$ **15** $f''(c)$ is $y'(c)$ **17** $\infty; .683, .749, .772 \rightarrow \frac{\pi}{4}$
19 $A + B + C = 1, \frac{1}{2}B + C = \frac{1}{2}, \frac{1}{4}B + C = \frac{1}{3}$; Simpson
21 $y = 1$ and x on [0,1]: $L_n = 1$ and $\frac{1}{2} - \frac{1}{2n}, R_n = 1$ and $\frac{1}{2} + \frac{1}{2n}$, so only $\frac{1}{2}L_n + \frac{1}{2}R_n$ gives 1 and $\frac{1}{2}$

23 $T_{10} \approx 500{,}000{,}000; T_{100} \approx 50{,}000{,}000; 25{,}000\pi$

25 $a = 4, b = 2, c = 1; \int_0^1 (4x^2 + 2x + 1)dx = \frac{10}{3}$; Simpson fits parabola **27** $c = \frac{1}{4320}$

2 The trapezoidal error has a factor $(\Delta x)^2$. It is reduced by 4 when Δx is cut in half. The error in Simpson's rule is proportional to $(\Delta x)^4$ and is reduced by 16.

4 Computing L_n and R_n requires n evaluations each. $T_n = \frac{1}{2}y_0 + y_1 + \cdots$ requires $n + 1$: more efficient.

8 The trapezoidal rule for $\int_0^{2\pi} \frac{dx}{3+\sin x} = \frac{\pi}{\sqrt{2}} = 2.221441$ gives $\frac{2\pi}{3} \approx 2.09$ (two intervals), $\frac{7\pi}{9} \approx 2.221$ (three intervals), $\frac{17\pi}{24} \approx 2.225$ (four intervals is worse??), and 7 digits for T_5. Curious that $M_n = T_n$ for odd n.

10 The midpoint rule is exact for 1 and x. For $y = x^2$ the integral from 0 to Δx is $\frac{1}{3}(\Delta x)^3$ and the rule gives $(\Delta x)(\frac{\Delta x}{2})^2$. This error $\frac{1}{4}(\Delta x)^3 - \frac{1}{3}(\Delta x)^3 = -\frac{1}{12}(\Delta x)^3$ does equal $-\frac{(\Delta x)^2}{24}(y'(\Delta x) - y'(0))$.

12 The first and third integrals give accurate answers more easily.

14 Correct answer $\frac{2}{3}$. $T_1 = .5, T_{10} \approx .66051, T_{100} \approx .66646$. $M_1 \approx .707, M_{10} \approx .66838, M_{100} \approx .66673$. What is the rate of decrease of the error?

16 $\int_{-1}^1 \frac{dx}{2+\cos 6\pi x} = \frac{2}{\sqrt{3}}$ is approximated by $T_2 = 1(\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}) = \frac{2}{3}$ and $S_2 = \frac{1}{6}(\frac{1}{3} + 4 \cdot \frac{1}{1} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{1} + \frac{1}{3}) = \frac{14}{9}$ and $G_1 = \frac{1}{2+\cos(-6\pi/\sqrt{3})} + \frac{1}{2+\cos(6\pi/\sqrt{3})} = .776$ (large error) and $G_2 = \frac{1}{2+\cos(6\pi\frac{1+1/\sqrt{3}}{2})} + \frac{1}{2+\cos(6\pi\frac{1-1/\sqrt{3}}{2})} \approx 1.5$.

18 The trapezoidal rule $T_4 = \frac{\pi}{8}(\frac{1}{2} + \cos^2\frac{\pi}{8} + \cos^2\frac{\pi}{4} + \cos^2\frac{3\pi}{8} + 0)$ gives the correct answer $\frac{\pi}{4}$.

20 $\frac{1}{90}(7y_0 + 32y_{1/4} + 12y_{1/2} + 32y_{3/4} + 7y_1)$ is correct over an interval for $y = 1, x, x^2, x^3, x^4$. Those five requirements give the five coefficients.

22 Any of these stopping points should give the integral as $0.886227 \cdots$ Extra correct digits depend on the computer design.

24 Directly $T_4 \approx 5.4248$. Separately on the intervals $[0, \pi]$ and $[\pi, 4]$, a single trapezoidal step T_1 is exact because $|x - \pi|$ is linear. Integral $= \frac{\pi^2}{2} + (8 - 4\pi + \frac{\pi^2}{2})$.

26 Simpson's rule gives $\frac{1}{6}(0^4 + 4(\frac{1}{2})^4 + 1^4) = \frac{5}{24}$. The difference from $\int_0^1 x^4 dx = \frac{1}{5}$ is $\frac{1}{120}$. Then $y'''(1) = 24$ and $y'''(0) = 0$ and $\frac{1}{120} = c(24)$ gives $c = \frac{1}{2880}$.

28 $y(a) = y(b)$.