CHAPTER 5 INTEGRALS

5.1 The Idea of the Integral (page 181)

The problem of summation is to add $v_1 + \cdots + v_n$. It is solved if we find f's such that $v_j = \mathbf{f_j - f_{j-1}}$. Then $v_1 + \cdots + v_n$ equals $\mathbf{f_n - f_0}$. The cancellation in $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$ leaves only $\mathbf{f_n}$ and $-\mathbf{f_0}$. Taking sums is the **reverse (or inverse)** of taking differences.

The differences between 0, 1, 4, 9 are $v_1, v_2, v_3 = \mathbf{1, 3, 5}$. For $f_j = j^2$ the difference between f_{10} and f_9 is $v_{10} = \mathbf{19}$. From this pattern $1 + 3 + 5 + \cdots + 19$ equals **100**.

For functions, finding the integral is the reverse of **finding the derivative.** If the derivative of $f(x)$ is $v(x)$, then the **integral** of $v(x)$ is $f(x)$. If $v(x) = 10x$ then $f(x) = \mathbf{5x^2}$. This is the **area** of a triangle with base x and height $10x$.

Integrals begin with sums. The triangle under $v = 10x$ out to $x = 4$ has area **80**. It is approximated by four rectangles of heights 10, 20, 30, 40 and area **100.** It is better approximated by eight rectangles of heights $5, 10, \cdots, 40$ and area **90.** For n rectangles covering the triangle the area is the sum of $\mathbf{\frac{4}{n}(\frac{40}{n} + \frac{80}{n} + \cdots + 40) = 80 + \frac{80}{n}}$. As $n \to \infty$ this sum should approach the number **80.** *That is the integral of* $v = 10x$ *from 0 to 4.*

1 $1, 3, 7, 15, 127$ **3** $-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8} - 1$ **5** $f_j - f_0 = \frac{r^j - 1}{r-1}$ **7** $3x$ for $x \le 7, 7x - 4$ for $x \ge 1$
9 $\frac{1}{52}\frac{1}{\sqrt{52}}, \frac{2}{52}, \frac{1}{52}\sqrt{\frac{j}{52}}$ **11** Lower by 2 **13** Up, down; rectangle **15** $\sqrt{x + \Delta x} - \sqrt{x}; \Delta x; \frac{df}{dx}; \sqrt{x}$
17 6; 18; triangle **19** 18 rectangles **21** $6x - \frac{1}{2}x^2 - 10; 6 - x$ **23** $\frac{14}{27}$ **25** $x^2; x^2; \frac{1}{3}x^3$

2 (a) $2^5 - 2^4 = 16 = v_5$ (b) $1 + 2 + 4 + 8 + 16 = f_5 - f_0 = \mathbf{31}$

4 Any C can be added to $f(x)$ because the **derivative** of a constant is zero.
Any C can be added to $f_0, f_1, \cdots$ because the **difference** between f's is not changed.

6 $f_0 = \frac{1-1}{r-1} = 0; 1 + r + \cdots + r^n = f_n = \mathbf{\frac{r^n - 1}{r-1}}$.

8 The f's are $\mathbf{0, 1, -1, 2, -2, \cdots}$ Here $v_j = (-1)^{j+1}j$ or $v_j = \begin{cases} j & j \text{ odd} \\ -j & j \text{ even} \end{cases}$ and $f_j = \begin{cases} \frac{j+1}{2} & j \text{ odd} \\ \frac{-j}{2} & j \text{ even} \end{cases}$

10 Within each quarter the sum over 13 weeks is lower than the single value for the whole quarter.

12 The last rectangle for the pessimist has height $\sqrt{\frac{15}{4}}$. Since the optimist's last rectangle of area $\frac{1}{4}\sqrt{\frac{16}{4}} = \frac{1}{2}$ is missed, the total area is reduced by $\frac{1}{2}$.

14 The optimist's rectangles contain the curve. The pessimist's rectangles lie under the curve.

16 Under the $\sqrt{x}$ curve, the first triangle has base 1, height 1, area $\frac{1}{2}$. To its right is a rectangle of area 3. Above the rectangle is a triangle of base 3, height 1, area $\frac{3}{2}$. The total area $\frac{1}{2} + 3 + \frac{3}{2} = 5$ is below the curve.

18 The total rectangular area is 21.

20 The rectangles have area 2 times 5, 2 times 3, and 2 times 1, adding to 18. This is exactly correct because each overestimate is compensated by an equal underestimate.

22 The region is a right triangle with height $6 - x$ and base $6 - x$ and area $\frac{1}{2}(6 - x)^2$. This has derivative $x - 6$, which is $-v(x)$ (minus sign because area decreases as x increases).

24 The areas under $\sqrt{x}$ and under x^2 add to 1. The same is true for the areas under x^3 and $x^{1/3}$.

Reason: Area under inverse function equals area above original function (provided $f(0) = 0$).
26 $A \approx 5.3313556$

5.2 Antiderivatives (page 186)

Integration yields the **area** under a curve $y = v(x)$. It starts from rectangles with the base $\mathbf{\Delta x}$ and heights $v(x)$ and areas $\mathbf{v(x)\Delta x}$. As $\Delta x \to 0$ the area $v_1\Delta x + \cdots + v_n\Delta x$ becomes the **integral** of $v(x)$. The symbol for the indefinite integral of $v(x)$ is $\int \mathbf{v(x)dx}$.

The problem of integration is solved if we find $f(x)$ such that $\mathbf{\frac{df}{dx} = v(x)}$. Then f is the **antiderivative** of v, and $\int_2^6 v(x)dx$ equals $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. The limits of integration are **2 and 6**. This is a **definite** integral, which is a **number** and not a function $f(x)$.

The example $v(x) = x$ has $f(x) = \mathbf{\frac{1}{2}x^2}$. It also has $f(x) = \mathbf{\frac{1}{2}x^2 + 1}$. The area under $v(x)$ from 2 to 6 is **16**. The constant is canceled in computing the difference $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. If $v(x) = x^8$ then $f(x) = \mathbf{\frac{1}{9}x^9}$.

The sum $v_1 + \cdots + v_n = f_n - f_0$ leads to the Fundamental Theorem $\int_a^b v(x)dx = \mathbf{f(b) - f(a)}$. The **indefinite** integral is $f(x)$ and the **definite** integral is $f(b) - f(a)$. Finding the **area** under the v-graph is the opposite of finding the **slope** of the f-graph.

1 $x^5 + \frac{2}{3}x^6; \frac{5}{3}$ **3** $2\sqrt{x}; 2$ **5** $\frac{3}{4}x^{4/3}(1+2^{1/3}); \frac{3}{4}(1+2^{1/3})$ **7** $-2\cos x - \frac{1}{2}\cos 2x; \frac{5}{2} - 2\cos 1 - \frac{1}{2}\cos 2$
9 $x\sin x + \cos x; \sin 1 + \cos 1 - 1$ **11** $\frac{1}{2}\sin^2 x; \frac{1}{2}\sin^2 1$ **13** $f = C; 0$ **15** $f(b) - f(a); f_8 - f_3$
17 $8 + \frac{8}{N}$ **19** $\frac{\pi}{3}(1+\sqrt{3}); \frac{\pi}{6}(3+\sqrt{3}); 2$ **21** $\frac{5}{2}; \frac{205}{36}; \infty$ **23** $f(x) = 2\sqrt{x}$ **25** $\frac{1}{2}$, below -1; $\frac{1}{4}, \frac{5}{4}$
27 Increase - decrease; increase - decrease - increase
29 Area under B − area under D; time when $B = D$; time when $B - D$ is largest **33** T; F; F; T; F

2 $f(x) = \frac{1}{2}x^2 + 4x^3; f(1) - f(0) = \mathbf{4\frac{1}{2}}$. **4** $f(x) = \frac{2}{5}x^{5/2}; f(1) - f(0) = \mathbf{\frac{2}{5}}$.
6 $\frac{x^{1/3}}{x^{2/3}} = x^{-1/3}$ which has antiderivative $f(x) = \frac{3}{2}x^{2/3}; f(1) - f(0) = \mathbf{\frac{3}{2}}$.
8 $f(x) = \tan x + x; f(1) - f(0) = \mathbf{\tan 1 + 1}$. **10** $f(x) = \sin x - x\cos x; f(1) - f(0) = \mathbf{\sin 1 - \cos 1}$
12 $f(x) = \frac{1}{3}\sin^3 x; f(1) - f(0) = \mathbf{\frac{1}{3}(\sin 1)^3}$.
14 $f(x) = -x$ plus any constant C; $f(1) - f(0) = -1 + C - C = \mathbf{-1}$.
16 The sum of v's is multiplied by $\mathbf{\Delta x}$. The difference of f's is divided by $\mathbf{\Delta x}$.
18 Areas 0, 1, 2, 3 add to $A_4 = \mathbf{6}$. Each rectangle misses a triangle of base $\frac{4}{N}$ and height $\frac{4}{N}$. There are N triangles of total area $N \cdot \frac{1}{2}(\frac{4}{N})^2 = \frac{8}{N}$. So the N rectangles have area $\mathbf{8 - \frac{8}{N}}$.
20 Example: Under $y = x^2$ the rectangles with heights $0, (.8)^2, (.9)^2$ and bases .8, .1, .1 have area .145. The two rectangles with heights 0 and $(.7)^2$ and bases .7 and .3 have larger area .147.
22 Two rectangles have base $\frac{1}{2}$ and heights 2 and 1, with area $\mathbf{\frac{3}{2}}$. Four rectangles have base $\frac{1}{4}$ and heights 4, 3, 2, 1 with area $\frac{10}{4} = \mathbf{\frac{5}{2}}$. Eight rectangles have area $\frac{7}{2}$. The limiting area under $y = \frac{1}{x}$ is **infinite.**
24 $\frac{1}{3}x^3$ is an antiderivative of x^2. So the area under x^2 from 0 to 4 is $\frac{1}{3}4^3 = \mathbf{\frac{64}{3}}$. The area under $\sqrt{x}$ is $\frac{16}{3}$. Those areas do **not** combine to give a rectangle.
26 Choose $v(x)$ to be **positive** until $x = 1$, **zero** to $x = 2$, then **negative** to $x = 3$. For total area 1,

take $v(x) = 2$ then 0 then -1.
28 The area $f(4) - f(3)$ is $-\frac{1}{2}$, and $f(3) - f(2)$ is -1, and $f(2) - f(1)$ is $\frac{1}{2}(\frac{2}{3})(2) - \frac{1}{2}(\frac{1}{3})(1)$. Total -1.
The graph of f_4 is x^2 to $x = 1$.
30 $y_4(x)$ equals 2 up to $x = 1$, then -3, then 0, then 1. **32** $12 =$ area of complete rectangle.

5.3 Summation Versus Integration (page 194)

The Greek letter $\sum$ indicates summation. In $\sum_1^n v_j$ the dummy variable is **j**. The limits are $\mathbf{j = 1}$ and $\mathbf{j = n}$, so the first term is $\mathbf{v_1}$ and the last term is $\mathbf{v_n}$. When $v_j = j$ this sum equals $\frac{1}{2}\mathbf{n(n+1)}$. For $n = 100$ the leading term is $\frac{1}{2}100^2 = 5000$. The correction term is $\frac{1}{2}\mathbf{n} = 50$. The leading term equals the integral of $v = x$ from 0 to 100, which is written $\int_0^{100} \mathbf{x\,dx}$. The sum is the total **area** of 100 rectangles. The correction term is the area between the **sloping line** and the **rectangles**.

The sum $\sum_{i=3}^6 i^2$ is the same as $\sum_{j=1}^4 (\mathbf{j+2})^2$ and equals **86**. The sum $\sum_{i=4}^5 v_i$ is the same as $\sum_{\mathbf{i=0}}^{\mathbf{1}} v_{i+4}$ and equals $\mathbf{v_4 + v_5}$. For $f_n = \sum_{j=1}^n v_j$ the difference $f_n - f_{n-1}$ equals $\mathbf{v_n}$.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n = \frac{1}{6}\mathbf{n(n+1)(2n+1)} = \frac{1}{3}\mathbf{n^3} + \frac{1}{2}\mathbf{n^2} + \frac{1}{6}\mathbf{n}$. To prove it by mathematical induction, check $f_1 = \mathbf{1}$ and check $f_n - f_{n-1} = \mathbf{n^2}$. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is $\frac{1}{3}\mathbf{9^3}$. This is close to the area of $\mathbf{9/\Delta x}$ rectangles of base Δx. The correction terms approach zero very **slowly**.

1 $\frac{25}{12}; 16$ **3** $127; 2^{n+1} - 1$ **5** $\sum_{j=1}^{50} 2j = 2550; \sum_{i=1}^{50}(2i-1) = 2500; \sum_{k=1}^{4}(-1)^{k+1}/k = \frac{7}{12}$
7 $\sum_{k=0}^{n} a_k x^k; \sum_{j=1}^{n} \sin\frac{2\pi j}{n}$ **9** 5.18738; 7.48547 **11** $2(a_i^2 + b_i^2)$ **13** $2^{n+1} - 1; \frac{1}{11} - \frac{1}{1}$ **15** F; T
17 $\frac{df}{dx} + C; f_9 - f_8 - f_1 + f_0$ **19** $f_1 = 1; n^2 + (2n+1) = (n+1)^2$
21 $a + b + c = 1, 2a + 4b + 8c = 5, 3a + 9b + 27c = 14$; sum of squares **23** $S_{400} = 80200; E_{400} = .0025 = \frac{1}{n}$
25 $S_{100,1/3} \approx 350, E_{100,1/3} \approx .00587; S_{100,3} = 25502500, E_{100,3} = .0201$ **27** v_1 and v_2 have the same sign
29 $v_1 = 9, v_2 = 12, \Sigma\Sigma = 21$ **31** At $N = 1, 2^{N-2}$ is not 1 **33** $0; \frac{1}{n}(v_1 + \cdots + v_n)$
35 $\Delta x \sum_{j=1}^{n} v(j\Delta x)$ **37** $f(1) - f(0) = \int_0^1 \frac{df}{dx}dx$

2 $8; 1 - \frac{1}{2^n}$ **4** The sums are $-1, 1, -2, 2, \cdots$ and the sum up to $n = 6$ is 3.
6 $\sum_{j=1}^4 (-1)^{j+1} v_j$; $\sum_{i=1}^n v_i w_i$; $\sum_{i=1}^3 v_{2i-1}$. **8** $(a+b)^n = \sum_{j=1}^n \binom{n}{j} a^{n-j} b^j$.
10 The first sum is close to $e^{-1} = .36788$; the second is close to $e = 2.71828$; the product is extremely near 1.
12 Choose all a's and b's equal to 1. Then $n^2 \neq n$.
14 $f_n - f_0$ and $f_{13} - f_3$ (by telescoping: the other terms cancel).
16 $\sum_{i=1}^n v_i = \sum_{j=0}^{n-1} \mathbf{v_{j+1}}$ and $\sum_{i=0}^6 i^2 = \sum_{i=2}^8 (\mathbf{i-2})^2$.

18 $f_1 = \frac{1}{6}(1)(2)(3) = 1; f_n - f_{n-1} = \frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}(n-1)(n)(2n-1) = n^2$.
20 $f_1 = \frac{1}{4}(1)^2(2)^2 = 1; f_n - f_{n-1} = \frac{1}{4}n^2(n+1)^2 - \frac{1}{4}(n-1)^2n^2 = \frac{1}{4}n^2(4n) = n^3$.
22 $q = \frac{1}{9}$ (emphasize the comparison with $\int x^8 dx = \frac{1}{9}x^9$).
24 $S_{50} = 42925; I_{50} = 41666\frac{2}{3}; D_{50} = 1258\frac{1}{3}; E_{50} = 0.0302$; E_n is approximately $\frac{1.5}{n}$ and exactly $\frac{1.5}{n} + \frac{1}{2n^2}$.
26 $E_{n,p} \approx \frac{p+1}{2n}$. Reason: A closer sum S includes only half of the last term n^p (trapezoidal rule: Section 5.8). Then $\frac{1}{2}n^p/I = \frac{p+1}{2n}$.
28 $xS = x + x^2 + x^3 + \cdots$ equals $S - 1$. Then $S = \frac{1}{1-x}$. If $x = 2$ the sums are $S = \infty$.
30 $(w_{2,1} + w_{2,2} + w_{2,3}); (w_{1,3} + w_{2,3})$; the sum is the same whether i or j comes first.
32 $4v_1 + 4v_2 + 4v_3 = 4(v_1 + v_2 + v_3); (u_1v_1 + u_1v_2 + u_1v_3) + (u_2v_1 + u_2v_2 + u_2v_3) = (u_1 + u_2)(v_1 + v_2 + v_3)$.
34 $14^2 = 196 \le (13)(17) = 221; (a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ because cancellation leaves $2a_1b_1a_2b_2 \le a_1^2b_2^2 + a_2^2b_1^2$ and this can be rewritten as $0 \le (a_1b_2 - a_2b_1)^2$ which is true.
36 The rectangular area is $\Delta x \sum_{j=1}^{1/\Delta x} v((j-1)\Delta x)$ or $\Delta x \sum_{i=0}^{(1/\Delta x)-1} v(i\Delta x)$.

5.4 Indefinite Integrals and Substitutions (page 200)

Finding integrals by substitution is the reverse of the **chain** rule. The derivative of $(\sin x)^3$ is $\mathbf{3(\sin x)^2 \cos x}$. Therefore the antiderivative of $\mathbf{3(\sin x)^2 \cos x}$ is $\mathbf{(\sin x)^3}$. To compute $\int(1 + \sin x)^2 \cos x\, dx$, substitute $u = \mathbf{1 + \sin x}$. Then $du/dx = \mathbf{\cos x}$ so substitute $du = \mathbf{\cos x\, dx}$. In terms of u the integral is $\int \mathbf{u^2\, du} = \mathbf{\frac{1}{3}u^3}$. Returning to x gives the final answer.

The best substitutions for $\int \tan(x+3)\sec^2(x+3)dx$ and $\int(x^2+1)^{10}x\, dx$ are $u = \mathbf{\tan(x+3)}$ and $u = \mathbf{x^2+1}$. Then $du = \mathbf{\sec^2(x+3)dx}$ and $\mathbf{2x\, dx}$. The answers are $\mathbf{\frac{1}{2}\tan^2(x+3)}$ and $\mathbf{\frac{1}{22}(x^2+1)^{11}}$. The antiderivative of $v\, dv/dx$ is $\mathbf{\frac{1}{2}v^2}$. $\int 2x\, dx/(1+x^2)$ leads to $\int \mathbf{\frac{du}{u}}$, which we don't yet know. The integral $\int dx/(1+x^2)$ is known immediately as $\mathbf{\tan^{-1}x}$.

1 $\frac{2}{3}(2+x)^{3/2} + C$ **3** $(x+1)^{n+1}/(n+1) + C(n \ne -1)$ **5** $\frac{1}{12}(x^2+1)^6 + C$ **7** $-\frac{1}{4}\cos^4 x + C$
9 $-\frac{1}{8}\cos^4 2x + C$ **11** $\sin^{-1} t + C$ **13** $\frac{1}{3}(1+t^2)^{3/2} - (1+t^2)^{1/2} + C$ **15** $2\sqrt{x} + x + C$
17 $\sec x + C$ **19** $-\cos x + C$ **21** $\frac{1}{3}x^3 + \frac{2}{3}x^{3/2}$ **23** $-\frac{1}{3}(1-2x)^{3/2}$ **25** $y = \sqrt{2x}$
27 $\frac{1}{2}x^2$ **29** $a\sin x + b\cos x$ **31** $\frac{4}{15}x^{5/2}$ **33** F; F; F; F **35** $f(x-1); 2f(\frac{x}{2})$
37 $x - \tan^{-1} x$ **39** $\int \frac{1}{u} du$ **41** $4.9t^2 + C_1 t + C_2$ **43** $f(t+3); f(t) + 3t; 3f(t); \frac{1}{3}f(3t)$

2 $\frac{-2}{3}(3-x)^{3/2} + C$ **4** $\frac{1}{1-n}(x+1)^{1-n}$, for $n \ne 1$. **6** $\frac{-2}{9}(1-3x)^{3/2} + C$ **8** $\frac{-1}{2\sin^2 x} + C$ or $-\frac{1}{2}(\sin x)^{-2} + C$
10 $\cos^3 x \sin 2x$ equals $2\cos^4 x \sin x$ and its integral is $\frac{-2}{5}\cos^5 x + C$ **12** $\frac{-1}{3}(1-t^2)^{3/2} + C$
14 Write $u = 1-t^2$ and $du = -2t\, dt$ to give $\int(1-u)\sqrt{u}\frac{du}{-2} = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C = -\frac{1}{3}(1-t^2)^{3/2} + \frac{1}{5}(1-t^2)^{5/2} + C$
16 The integral of $x^{1/2} + x^2$ is $\frac{2}{3}x^{3/2} + \frac{1}{3}x^3 + C$.
18 Set $u = \tan x$ and $du = \sec^2 x\, dx$. The integral of $u^2 du$ is $\frac{1}{3}\tan^3 x + C$.
20 Write $\sin^3 x$ as $(1 - \cos^2 x)\sin x$. The integrals of $-\cos^2 x \sin x$ and $\sin x$ give $\frac{1}{3}\cos^3 x - \cos x + C$.
22 Substitute $y = cx^n$ to find $ncx^{n-1} = (cx^n)^2$. Match exponents: $n - 1 = 2n$ or $n = -1$. Match coefficients: $nc = c^2$ or $c = n = -1$. Answer $y = \mathbf{-1/x}$.
24 $y = -\sqrt{1-2x} + C$ **26** $dy/dx = x/y$ gives $y\, dy = x\, dx$ or $y^2 = x^2 + C$ or $y = \sqrt{x^2 + C}$.