43 Decreasing; $\frac{dx}{dy} = \frac{1}{dy/dx} < 0$ **45** F; T; F **47** $g(x) = x^m, f(y) = y^n, x = (z^{1/n})^{1/m}$
49 $g(x) = x^3, f(y) = y + 6, x = (z-6)^{1/3}$ **51** $g(x) = 10^x, f(y) = \log y, x = \log(10^y) = y$
53 $y = x^3, y'' = 6x, d^2x/dy^2 = -\frac{2}{9}y^{-5/3}$; m/ sec^2, sec /m^2 **55** $p = \frac{1}{\sqrt{y}} - 1; 0 < y \le 1$
57 $\max = G = \frac{3}{8}y^{4/3}, G' = \frac{1}{2}y^{1/3}$ **59** $y^2/100$

2 $x = \frac{y-B}{A}$ **4** $x = \frac{y}{y-1}$(f^{-1} matches f) **6** no inverse **8** $x = \begin{cases} \frac{1}{3}y & y \ge 0 \\ y & y \le 0 \end{cases}$ **10** $x = y^5$
12 The graph is a hyperbola, **symmetric across the 45° line;** $\frac{dy}{dx} = -\frac{2}{(x-1)^2}; \frac{dx}{dy} = -\frac{1}{2}(x-1)^2$ (or $-\frac{2}{(y-1)^2}$).
14 f^{-1} does not exist because $f(3)$ is the same as $f(5)$.
16 No two x's give the same y. **18** $y = \frac{x}{x-1}$ and $y = 2 - x$ (functions of $x + y$ and xy lead to suitable f)
20 The inverse of a piecewise linear function is piecewise **linear** (if the inverse exists).
22 $\frac{dy}{dx} = -\frac{1}{(x-1)^2}; \frac{dx}{dy} = -\mathbf{\frac{1}{y^2}} = -(x-1)^2$. **24** $\frac{dy}{dx} = -\frac{3}{x^4}; \frac{dx}{dy} = -\mathbf{\frac{1}{3}y^{-4/3}}$. **26** $\frac{dy}{dx} = \frac{ad-bc}{(cx+d)^2}; \frac{dx}{dy} = \mathbf{\frac{ad-bc}{(cy-a)^2}}$.
28 $\frac{dy}{dx} = y$. **30** jumps at $0, y_1, y_2$ to heights x_1, x_2, x_3; a piecewise constant function has no inverse.
32 Hyperbola centered at $(-1, 0)$: shift the standard hyperbola $xy = 1$.
34 $y = -3x$ for $x \le 0; y = -x$ for $x \ge 0$. **36** The graph is the first quarter of the unit circle.
38 The graph starts at (0,1) and increases with vertical asymptote at $x = 1$.
40 $1 = \sec^2 x \frac{dx}{dy}$ so $\frac{dx}{dy} = \cos^2 x = \mathbf{\frac{1}{2}}$ **42** $\frac{dy}{dx} = 1 - \cos x = 0$ so $\frac{dx}{dy} = \infty$. (The derivative does not exist.)
44 **First proof** Suppose $y = f(x)$. We are given that $y > x$. This is the same as $y > f^{-1}(y)$.
Second proof The graph of $f(x)$ is above the 45° line, because $f(x) > x$. The mirror image is below the 45° line so $f^{-1}(y) < y$.
46 $g(x) = x - 4, f(y) = 5y, g^{-1}(y) = y + 4, f^{-1}(z) = \frac{z}{5}, \mathbf{x = \frac{1}{5}z + 4}$.
48 $g(x) = x + 6, f(y) = y^3, g^{-1}(y) = y - 6, f^{-1}(z) = \sqrt[3]{z}; \mathbf{x = \sqrt[3]{z} - 6}$
50 $g(x) = \frac{1}{2}x + 4, f(y) = g(y), g^{-1}(y) = 2y - 8, f^{-1}(z) = g^{-1}(z); \mathbf{x = 2(2z - 8) - 8 = 4z - 24}$.
52 $x^* = f^{-1}(0)$
54 $f^{-1}(0) \approx f^{-1}(y) + (\frac{df^{-1}}{dy})(0 - y)$ is the same as $x^* \approx x + \frac{1}{df/dx}(0 - f(x))$, which gives Newton's method.
56 $\frac{dG}{dy} = f^{-1}(y) + y\frac{df^{-1}}{dy} - F'(f^{-1}(y))\frac{df^{-1}}{dy}$. The second term cancels the third because $F'(f^{-1}(y))$ is equal to $f(f^{-1}(y)) = y$. This leaves the first term $\frac{dG}{dy} = f^{-1}(y)$. **G is the antiderivative of $\mathbf{f^{-1}}$ if $\mathbf{F' = f}$.**
58 To maximize $yx - F(x)$ set the x derivative to zero: $y = \frac{dF}{dx} = f(x)$ or $x = f^{-1}(y)$. Substitute this x into $xy - F(x)$: the maximum value is exactly $G(y)$ from Problem 56. Now maximize $xy - G(y)$. The y derivative gives $x = \frac{dG}{dy}$ or by Problem 56 $x = f^{-1}(y)$. Substitute $y = f(x)$ into $xy - G(y)$ to find that the maximum value is $xf(x) - G(f(x)) = xf(x) - [f(x)x - F(f^{-1}(f(x))] = F(x)$.
Note: This is the **Legendre transform** between $F(x)$ and $G(y)$ – important but not well known. Since $\frac{dF}{dx}$ is increasing (then f^{-1} exists), the function $F(x)$ is convex (concave up). So is $G(y)$.

4.4 Inverses of Trigonometric Functions (page 175)

The relation $x = \sin^{-1} y$ means that **y** is the sine of **x**. Thus x is the angle whose sine is **y**. The number y lies between **−1** and **1**. The angle x lies between $-\pi/2$ and $\pi/2$. (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle $\sin^{-1} y$ is $\sqrt{\mathbf{1 - y^2}}$. The derivative of $x = \sin^{-1} y$ is

$dx/dy = \mathbf{1/\sqrt{1-y^2}}$.

The relation $x = \cos^{-1} y$ means that y equals **cos x**. Again the number y lies between **−1** and **1**. This time the angle x lies between **0** and π (so that each y comes from only one angle x). The sum $\sin^{-1} y + \cos^{-1} y = \pi/2$. (The angles are called **complementary**, and they add to a **right** angle.) Therefore the derivative of $x = \cos^{-1} y$ is $dx/dy = \mathbf{-1/\sqrt{1-y^2}}$, the same as for $\sin^{-1} y$ except for a **minus** sign.

The relation $x = \tan^{-1} y$ means that $y = \mathbf{\tan x}$. The number y lies between $-\infty$ and ∞. The angle x lies between $-\pi/2$ and $\pi/2$. The derivative is $dx/dy = \mathbf{1/(1+y^2)}$. Since $\tan^{-1} y + \cot^{-1} y = \pi/2$, the derivative of $\cot^{-1} y$ is the same except for a **minus** sign.

The relation $x = \sec^{-1} y$ means that $\mathbf{y = \sec x}$. The number y *never* lies between **−1** and **1**. The angle x lies between **0** and π, but never at $x = \pi/2$. The derivative of $x = \sec^{-1} y$ is $dx/dy = \mathbf{1/\,|\,y\,|\,\sqrt{y^2-1}}$.

1 $0, \frac{\pi}{2}, 0$ **3** $\frac{\pi}{2}, 0, \frac{\pi}{4}$ **5** π is outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$ **7** $y = -\sqrt{3}/2$ and $\sqrt{3}/2$
9 $\sin x = \sqrt{1-y^2}; \sqrt{1-y^2}$ and 1 **11** $\frac{d(\sin^{-1} y)}{dy} \cos x = 1 \rightarrow \frac{d(\sin^{-1} y)}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$
13 $y = 0: 1, -1, 1; y = 1: 0, 0, \frac{1}{2}$ **15** F; F; T; T; F; F **17** $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ **19** $\frac{dz}{dx} = 3$
21 $\frac{dz}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$ **23** $1 - \frac{y \sin^{-1} y}{\sqrt{1-y^2}}$ **25** $\frac{dx}{dy} = \frac{1}{|y+1|\sqrt{y^2+2y}}$ **27** $u = 1$ so $\frac{du}{dy} = 0$ **31** $\sec x = \sqrt{y^2+1}$
33 $\frac{1}{10}, 1, \frac{1}{2}$ **35** $-y/\sqrt{1-y^2}$ **37** $\frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2}$ **39** $\frac{nx^{n-1}}{|x^n|\sqrt{x^{2n}-1}}$ **41** $\frac{dy}{dx} = \frac{1}{1+x^2}$
43 $\frac{dy}{dx} = \frac{1}{1+x^2}$ **47** $u = 4\sin^{-1} y$ **49** π **51** $-\pi/4$

2 $\sin^{-1}(-1) = -\frac{\pi}{2}; \cos^{-1}(-1) = \pi; \tan^{-1}(-1) = -\frac{\pi}{4}$. Note that $-\frac{\pi}{2}, \pi, -\frac{\pi}{4}$ are in the required ranges.
4 $\sin^{-1}\sqrt{3}$ doesn't exist; $\cos^{-1}\sqrt{3}$ doesn't exist; $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.
6 The range of $\sin^{-1}(y)$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Note that $\sin 2\pi = 0$ but 2π is not $\sin^{-1} 0$.
8 $\frac{dx}{dy} = \frac{1}{2\sqrt{1-y^2/4}} = \frac{1}{\sqrt{4-y^2}}$. The graph goes from $y = -\pi$ to $y = \pi$.
10 The sides of the triangle are $y, \sqrt{1-y^2}$, and 1. The tangent is $\mathbf{\frac{y}{\sqrt{1-y^2}}}$.
12 $\frac{d\sin^{-1}}{dy}(\sin x)\cos x$ equals $\frac{1}{\sqrt{1-\sin^2 x}} \cos x = \mathbf{1}$ as required.
14 $\frac{d(\sin^{-1} y)}{dy}|_{x=0} = \mathbf{1}; \frac{d(\cos^{-1} y)}{dy}|_{x=0} = -\infty; \frac{d(\tan^{-1} y)}{dy}|_{x=0} = \mathbf{1}; \frac{d(\sin^{-1} y)}{dy}|_{x=1} = \mathbf{\frac{1}{\cos 1}}; \frac{d(\cos^{-1} y)}{dy}|_{x=1} = -\mathbf{\frac{1}{\sin 1}};$ $\frac{d(\tan^{-1} y)}{dy}|_{x=1} = \mathbf{\frac{1}{\sec^2 1}}$.
16 $\cos^{-1}(\sin x)$ is the complementary angle $\frac{\pi}{2} - x$. The tangent of that angle is $\mathbf{\frac{\cos x}{\sin x} = \cot x}$.
18 $\frac{du}{dx} = \frac{1}{1+(2x)^2}(2) = \mathbf{\frac{2}{1+4x^2}}$. **20** $\frac{du}{dx} = \frac{1}{\sqrt{1-(\cos x)^2}}(-\sin x) = \mathbf{-1}$. Check: $z = \frac{\pi}{2} - x$ so $\frac{dz}{dx} = -1$.
22 $\frac{dz}{dx} = -1(\sin^{-1} x)^{-2}\frac{1}{\sqrt{1-x^2}}$. **24** $\frac{dz}{dx} = 2x\tan^{-1} x + (1+x^2)\frac{1}{1+x^2} = \mathbf{2x\tan^{-1} x + 1}$.
26 $u = x^2$ so $\frac{du}{dx} = \mathbf{2x}$. **28** $\frac{du}{dy} = \mathbf{\frac{1}{1+y^2}}$. The range of this function is $0 \le y \le \frac{\pi}{2}$.
30 The right triangle has far side y and near side 1. Then the near angle is $\tan^{-1} y$. That angle is also $\cot^{-1}(\frac{1}{y})$.
34 The requirement is $u' = \frac{1}{1+t^2}$. To satisfy this requirement take $u = \mathbf{\tan^{-1} t}$.
36 $u = \tan^{-1} y$ has $\frac{du}{dy} = \frac{1}{1+y^2}$ and $\frac{d^2u}{dy^2} = \mathbf{\frac{-2y}{(1+y^2)^2}}$. **38** $\frac{du}{dy} = \frac{2}{|2y|\sqrt{(2y)^2-1}} = \frac{1}{|y|\sqrt{4y^2-1}}$.

40 By the chain rule $\frac{du}{dx} = \frac{1}{|\tan x|\sqrt{\tan^2 x - 1}}(\sec^2 x)$.

42 By the product rule $\frac{dz}{dx} = (\cos x)(\sin^{-1} x) + (\sin x)\frac{1}{\sqrt{1-x^2}}$. Note that $z \neq x$ and $\frac{dz}{dx} \neq 1$.

44 $\frac{dz}{dx} = \cos(\cos^{-1} x)(\frac{-1}{\sqrt{1-x^2}}) + \sin(\sin^{-1} x)(\frac{1}{\sqrt{1-x^2}}) = \frac{-x+x}{\sqrt{1-x^2}} = \mathbf{0}$.

46 Domain $|y| \geq 1$; range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ with $x = 0$ deleted.

48 $u(x) = \mathbf{\frac{1}{2}\tan^{-1} 2x}$ (need $\frac{1}{2}$ to cancel 2 from the chain rule).

50 $u(x) = \frac{x-1}{x+1}$ has $\frac{du}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. Then $\frac{d}{dx}\tan^{-1} u(x) = \frac{1}{1+u^2}\frac{du}{dx} = \frac{1}{1+(\frac{x-1}{x+1})^2}\frac{2}{(x+1)^2} = \frac{2}{(x+1)^2+(x-1)^2} = \mathbf{\frac{1}{x^2+1}}$. This is also the derivative of $\mathbf{\tan^{-1} x}$! So $\tan^{-1} u(x)$ minus $\tan^{-1} x$ is a **constant**.

52 Problem 51 finds $u(0) = -1$ and $\tan^{-1} u(0) = -\frac{\pi}{4}$ and $\tan^{-1} 0 = 0$ and therefore $\tan^{-1} u(x) - \tan^{-1} x$ should have the constant value $-\frac{\pi}{4} - 0$. But as $x \to -\infty$ we now find $u \to 1$ and $\tan^{-1} u \to \frac{\pi}{4}$ and the difference is $\frac{\pi}{4} - (-\frac{\pi}{2}) = \frac{3\pi}{4}$. **The "constant" has changed!** It happened when x passed -1 and u became infinite and the angle $\tan^{-1} u$ jumped.