CHAPTER 4 DERIVATIVES BY THE CHAIN RULE

4.1 The Chain Rule (page 158)

$z = f(g(x))$ comes from $z = f(y)$ and $y = \mathbf{g(x)}$. At $x = 2$ the chain $(x^2-1)^3$ equals $\mathbf{3^3 = 27}$. Its inside function is $y = \mathbf{x^2 - 1}$, its outside function is $z = \mathbf{y^3}$. Then dz/dx equals $\mathbf{3y^2 dy/dx}$. The first factor is evaluated at $y = \mathbf{x^2 - 1}$ (not at $y = x$). For $z = \sin(x^4 - 1)$ the derivative is $\mathbf{4x^3 \cos(x^4 - 1)}$. The triple chain $z = \mathbf{\cos(x+1)^2}$ has a shift and a **square** and a cosine. Then $dz/dx = \mathbf{2\cos(x+1)(-\sin(x+1))}$.

The proof of the chain rule begins with $\Delta z/\Delta x = \mathbf{(\Delta z/\Delta y)(\Delta y/\Delta x)}$ and ends with $\mathbf{dz/dx = (dz/dy)(dy/dx)}$. Changing letters, $y = \cos u(x)$ has $dy/dx = \mathbf{-\sin u(x) \frac{du}{dx}}$. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx = \mathbf{nu^{n-1}\frac{du}{dx}}$. The slope of $5g(x)$ is $\mathbf{5g'(x)}$ and the slope of $g(5x)$ is $\mathbf{5g'(5x)}$. When $f =$ cosine and $g =$ sine and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are **1 and sin 1 and 0**.

1 $z = y^3, y = x^2 - 3, z' = 6x(x^2-3)^2$ **3** $z = \cos y, y = x^3, z' = -3x^2 \sin x^3$
5 $z = \sqrt{y}, y = \sin x, z' = \cos x / 2\sqrt{\sin x}$ **7** $z = \tan y + (1/\tan x), y = 1/x, z' = (\frac{-1}{x^2})\sec^2(\frac{1}{x}) - (\tan x)^{-2}\sec^2 x$
9 $z = \cos y, y = x^2 + x + 1, z' = -(2x+1)\sin(x^2+x+1)$ **11** $17\cos 17x$ **13** $\sin(\cos x)\sin x$
15 $x^2\cos x + 2x\sin x$ **17** $(\cos\sqrt{x+1})\frac{1}{2}(x+1)^{-1/2}$ **19** $\frac{1}{2}(1+\sin x)^{-1/2}(\cos x)$ **21** $\cos(\frac{1}{\sin x})(\frac{-\cos x}{\sin^2 x})$
23 $8x^7 = 2(x^2)^2(2x^2)(2x)$ **25** $2(x+1) + \cos(x+\pi) = 2x + 2 - \cos x$
27 $(x^2+1)^2 + 1$; $\sin U$ from 0 to sin 1; $U(\sin x)$ is 1 and 0 with period 2π; R from 0 to x; $R(\sin x)$ is half-waves.
29 $g(x) = x + 2, h(x) = x^2 + 2; k(x) = 3$ **31** $f'(f(x))f'(x)$; no; $(-1/(1/x)^2)(-1/x^2) = 1$ and $f(f(x)) = x$
33 $\frac{1}{2}(\frac{1}{2}x + 8) + 8; \frac{1}{8}x + 14; \frac{1}{16}$ **35** $f(g(x)) = x, g(f(y)) = y$
37 $f(g(x)) = \frac{1}{1-x}, g(f(x)) = 1 - \frac{1}{x}, f(f(x)) = x = g(g(x)), g(f(g(x))) = \frac{x}{x-1} = f(g(f(x)))$
39 $f(y) = y - 1, g(x) = 1$ **43** $2\cos(x^2+1) - 4x^2\sin(x^2+1); -(x^2-1)^{-3/2}; -(\cos\sqrt{x})/4x + (\sin\sqrt{x})/4x^{3/2}$
45 $f'(u(t))u'(t)$ **47** $(\cos^2 u(x) - \sin^2 u(x))\frac{du}{dx}$ **49** $2xu(x) + x^2\frac{du}{dx}$ **51** $1/4\sqrt{1-\sqrt{1-x}}\sqrt{1-x}$
53 df/dt **55** $f'(g(x))g'(x) = 4(x^3)^3 3x^2 = 12x^{11}$ **57** $3600; \frac{1}{2}; 18$ **59** $3; \frac{1}{3}$

2 $f(y) = y^2; g(x) = x^3 - 3; \frac{dz}{dx} = 6x^2(x^3-3)$ **4** $f(y) = \tan y; g(x) = 2x; \frac{dz}{dx} = 2\sec^2 2x$
6 $f(y) = \sin y; g(x) = \sqrt{x}; \frac{dz}{dx} = \frac{\cos x}{2\sqrt{x}}$ **8** $f(y) = \sin y; g(x) = \cos x; \frac{dz}{dx} = -\sin x\cos(\cos x)$
10 $f(y) = \sqrt{y}; g(x) = x^2; \frac{dz}{dx} = (\frac{1}{2\sqrt{y}})(2x) = 1$ **12** $\frac{dz}{dx} = \sec^2(x+1)$ **14** $\frac{dz}{dx} = 3x^2$ **16** $\frac{dz}{dx} = \frac{27}{2}\sqrt{9x+4}$
18 $\frac{dz}{dx} = \frac{\cos(x+1)}{2\sqrt{\sin(x+1)}}$ **20** $\frac{dz}{dx} = \frac{\cos(\sqrt{x}+1)}{2\sqrt{x}}$ **22** $\frac{dz}{dx} = 4x(\sin x^2)(\cos x^2)$
24 $\frac{dz}{dx} = \mathbf{3(3x)^2(3)}$ or $z = 27x^3$ and $\frac{dz}{dx} = \mathbf{81x^2}$ **26** $\frac{dz}{dx} = \frac{2\cos x\sin x}{2\sqrt{1-\cos^2 x}} = \mathbf{\cos x}$ or $z = \sin x$ and $\frac{dz}{dx} = \mathbf{\cos x}$
28 $f(y) = y + 1; h(y) = \sqrt[3]{y}; k(y) \equiv 1$ **30** $f(y) = \sqrt{y}, g(x) = 1 - x^2; f(y) = \sqrt{1-y}, g(x) = x^2$
32 (a) 22 (b) $4f'(5)$ (c) 8 (d) 4 **34** $C = \mathbf{16}$ because this solves $C = \frac{1}{2}C + 8$ (fixed point)
36 $f(y), g(x), |f(g(x)) - 9| < \epsilon$
38 For $g(g(x)) = x$ the graph of g should be **symmetric across the 45° line**: If the point (x, y) is on the graph so is (y, x). Examples: $g(x) = -\frac{1}{x}$ or $-x$ or $\sqrt[3]{1-x^3}$.

40 False (The chain rule produces -1 : so derivatives of even functions are odd functions)
False (The derivative of $f(x) = x$ is $f'(x) = 1$) **False** (The derivative of $f(1/x)$ is $f'(1/x)$ times $-1/x^2$)
True (The factor from the chain rule is 1) **False** (see equation (8)).

42 From $x = \frac{\pi}{4}$ go up to $y = \sin\frac{\pi}{4}$. Then go **across** to the parabola $z = y^2$. Read off $z = \sqrt{\sin\frac{\pi}{4}}$ on the horizontal z axis.

44 This is the chain rule applied to $\frac{dz}{dy}$ (a function of y). Its x derivative is its y derivative $(\frac{d^2z}{dy^2})$ times $\frac{dy}{dx}$.
If $z = y^2$ and $y = x^3$ then $\frac{dz}{dy} = 2y$ and $\frac{d^2z}{dy^2}\frac{dy}{dx} = \mathbf{2(3x^2)}$. Check another way: $\frac{dz}{dy} = 2x^3$ and $\frac{d}{dx}(\frac{dz}{dy}) = \mathbf{6x^2}$.

46 $\frac{dz}{dx} = (3u^2)(3x^2) = 9x^8$ **48** $\frac{dy}{dt} = \frac{1}{2\sqrt{u(t)}}\frac{du}{dt}$ **50** $\frac{dy}{dx} = 2xf'(x^2) + 2f(x)\frac{df}{dx}$

52 $\frac{dz}{dt} = -nu(t)^{-n-1}\frac{du}{dt}$ **54** $\frac{dy}{dt} = -\frac{1}{t^2}$ **56** $\cos(\sin x)\cos x$

58 (a) 53 (sum rule for derivatives) (b) 60 (chain rule)

60 Note that $G' = \cos(\sin x)\cos x$ and $G'' = -\cos(\sin x)\sin x - \sin(\sin x)\cos^2 x$. We were told that $H(x) = \cos(\cos x)$ should be included too.

4.2 Implicit Differentiation and Related Rates (page 163)

For $x^3 + y^3 = 2$ the derivative dy/dx comes from **implicit** differentiation. We don't have to solve for **y**. Term by term the derivative is $3x^2 + \mathbf{3y^2\frac{dy}{dx}} = 0$. Solving for dy/dx gives $\mathbf{-x^2/y^2}$. At $x = y = 1$ this slope is $\mathbf{-1}$. The equation of the tangent line is $y - 1 = \mathbf{-1(x-1)}$.

A second example is $y^2 = x$. The x derivative of this equation is $\mathbf{2y\frac{dy}{dx}} = 1$. Therefore $dy/dx = \mathbf{1/2y}$. Replacing y by $\sqrt{x}$ this is $dy/dx = \mathbf{1/2\sqrt{x}}$.

In related rates, we are given dg/dt and we want df/dt. We need a relation between f and **g**. If $f = g^2$, then $(df/dt) = \mathbf{2g}(dg/dt)$. If $f^2 + g^2 = 1$, then $df/dt = \mathbf{-\frac{g}{f}\frac{dg}{dt}}$. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt = \mathbf{3s^2(2)} = \mathbf{6s^2}$. To find a number (8 is wrong), you also need to know **s**.

1 $-x^{n-1}/y^{n-1}$ **3** $\frac{dy}{dx} = 1$ **5** $\frac{dy}{dx} = \frac{1}{F'(y)}$ **7** $(y^2 - 2xy)/(x^2 - 2xy)$ or 1 **9** $\frac{1}{\sec^2 y}$ or $\frac{1}{1+x^2}$

11 First $\frac{dy}{dx} = -\frac{y}{x}$, second $\frac{dy}{dx} = \frac{x}{y}$ **13** Faster, faster **15** $2zz' = 2yy' \to z' = \frac{y}{x}y' = y'\sin\theta$

17 $\sec^2\theta = \frac{c}{200\pi}$ **19** $500\frac{df}{dx}; 500\sqrt{1 + (\frac{df}{dx})^2}$ **21** $\frac{dy}{dt} = -\frac{8}{3}; \frac{dy}{dt} = -2\sqrt{3}; \infty$ then 0

23 $V = \pi r^2 h; \frac{dh}{dt} = \frac{1}{4\pi}\frac{dV}{dt} = -\frac{1}{4\pi}$ in/sec **25** $A = \frac{1}{2}ab\sin\theta, \frac{dA}{dt} = 7$ **27** 1.6 m/sec; 9 m/sec; 12.8 m/sec

29 $-\frac{7}{5}$ **31** $\frac{dz}{dt} = \frac{\sqrt{2}}{2}\frac{dy}{dt}; \frac{d\theta}{dt} = \frac{1}{10}\cos^2\theta\frac{d\theta}{dt}; \theta'' = \frac{\cos\theta}{10}y'' - \frac{1}{50}\cos^3\theta\sin\theta(y')^2$

2 $\frac{dy}{dx} = -\mathbf{\frac{y^2+2xy}{x^2+2xy}}$ **4** $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = \mathbf{-\frac{1}{2}}$ **6** $f'(x) + F'(y)\frac{dy}{dx} = y + x\frac{dy}{dx}$ so $\frac{dy}{dx} = \mathbf{\frac{y-f'(x)}{F'(y)-x}}$

8 $1 = \cos y\frac{dy}{dx}$ so $\frac{dy}{dx} = \mathbf{\frac{1}{\cos y}} = \mathbf{\frac{1}{\sqrt{1-x^2}}}$. **10** $ny^{n-1}\frac{dy}{dx} = 1$ so $\frac{dy}{dx} = \mathbf{\frac{1}{n}}$

12 $2(x-2) + 2y\frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = \mathbf{1}$ at (1,1); $2x + 2(y-2)\frac{dy}{dx} = 0$ also gives $\frac{dy}{dx} = \mathbf{1}$.

14 $2 + 2y\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^2 = 0$ yields $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{x^2}{y^3} = -\mathbf{\frac{y^2+x^2}{y^3}}$.

16 y catches up to z as θ increases to $\frac{\pi}{2}$. So y' should be larger than z'. **18** y' approaches $200\pi c/200\pi = \mathbf{c}$

20 x is a constant (fixed at 7) and therefore **a change Δ x is not allowed**

22 $x^2+y^2 = 10^2$ so $2x\frac{dx}{dt}+2y\frac{dy}{dt} = 0$ and $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -2\frac{x}{y} = -c$ when $x = \frac{1}{2}cy$. This means $(\frac{1}{2}cy)^2+y^2 = 10^2$ or $y = \frac{\mathbf{10}}{\sqrt{\mathbf{1+(\frac{1}{2}c)^2}}}$

24 Distance to you is $\sqrt{x^2+8^2}$, rate of change is $\frac{x}{\sqrt{x^2+8^2}}\frac{dx}{dt}$ with $\frac{dx}{dt} = 560$. (a) Distance $= 16$ and $x = 8\sqrt{3}$ and rate is $\frac{8\sqrt{3}}{16}(560) = \mathbf{280\sqrt{3}}$; (b) $x = 8$ and rate is $\frac{8}{\sqrt{8^2+8^2}}(560) = \mathbf{280\sqrt{2}}$; (c) $x = 0$ and rate is **zero.**

26 $10c(t-3) = 8t$ divided by $c(t-3) = 4$ gives $10 = 2t$. So $\mathbf{t = 5}$ and $\mathbf{c = 2}$. The x and y distances between ball and receiver are $2t-10$ and $12t-60$. The derivative of $\sqrt{(2t-10)^2+(12t-60)^2}$ $= \sqrt{148}|t-5|$ is $-\sqrt{\mathbf{148}}$.

28 Volume $= \frac{4}{3}\pi r^3$ has $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. If this equals twice the surface area $4\pi r^2$ (with minus for evaporation) than $\frac{dr}{dt} = \mathbf{-2}$.

30 $\frac{d\theta}{dt} = 4\pi$ radians/second; $0 = 2x\frac{dx}{dt} - 6\cos\theta\frac{dx}{dt} + 6x\sin\theta\frac{d\theta}{dt}$; at $\theta = \frac{\pi}{2}, x = 3\sqrt{3}$ and $6\sqrt{3}\frac{dx}{dt} + 18\sqrt{3}\frac{d\theta}{dt}$ gives $\frac{dx}{dt} = \mathbf{-12\pi}$; at $\theta = \pi, x = 0$ and $\frac{dx}{dt} = \mathbf{0}$.

4.3 Inverse Functions and Their Derivatives (page 170)

The functions $g(x) = x-4$ and $f(y) = y+4$ are **inverse** functions, because $f(g(x)) = \mathbf{x}$. Also $g(f(y)) = \mathbf{y}$. The notation is $f = g^{-1}$ and $g = \mathbf{f^{-1}}$. The composition **of f and $\mathbf{f^{-1}}$** is the identity function. By definition $x = g^{-1}(y)$ if and only if $y = \mathbf{g(x)}$. When y is in the range of g, it is in the **domain** of g^{-1}. Similarly x is in the **domain** of g when it is in the **range** of g^{-1}. If g has an inverse then $g(x_1)\neq g(x_2)$ at any two points. The function g must be steadily **increasing** or steadily **decreasing**.

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)(\mathbf{dg/dx}) = \mathbf{1}$. The slope of g^{-1} times the slope of g equals **1**. More directly $dx/dy = 1/(\mathbf{dy/dx})$. For $y = 2x+1$ and $x = \frac{1}{2}(y-1)$, the slopes are $dy/dx = \mathbf{2}$ and $dx/dy = \mathbf{\frac{1}{2}}$. For $y = x^2$ and $x = \sqrt{\mathbf{y}}$, the slopes are $dy/dx = \mathbf{2x}$ and $dx/dy = \mathbf{1/2\sqrt{y}}$. Substituting x^2 for y gives $dx/dy = \mathbf{1/2x}$. Then $(dx/dy)(dy/dx) = \mathbf{1}$.

The graph of $y = g(x)$ is also the graph of $x = \mathbf{g^{-1}(y)}$, but with x across and y up. For an ordinary graph of g^{-1}, take the reflection in the line $y = x$. If (3,8) is on the graph of g, then its mirror image **(8,3)** is on the graph of g^{-1}. Those particular points satisfy $8 = 2^3$ and $3 = \mathbf{\log_2 8}$.

The inverse of the chain $z = h(g(x))$ is the chain $x = \mathbf{g^{-1}(h^{-1}(z))}$. If $g(x) = 3x$ and $h(y) = y^3$ then $z = \mathbf{(3x)^3} = \mathbf{27x^3}$. Its inverse is $x = \mathbf{\frac{1}{3}z^{1/3}}$, which is the composition of $\mathbf{g^{-1}(y)} = \mathbf{\frac{1}{3}y}$ and $\mathbf{h^{-1}(z)} = \mathbf{z^{1/3}}$.

1 $x = \frac{y+6}{3}$ **3** $x = \sqrt{y+1}$ (x unrestricted $\to$ no inverse) **5** $x = \frac{y}{y+1}$ **7** $x = (1+y)^{1/3}$

9 (x unrestricted $\to$ no inverse) **11** $y = \frac{1}{x-a}$ **13** $2 < f^{-1}(x) < 3$ **15** f goes up and down

17 $f(x)g(x)$ and $\frac{1}{f(x)}$ **19** $m\neq 0; m\geq 0; |m|\geq 1$ **21** $\frac{dy}{dx} = 5x^4, \frac{dx}{dy} = \frac{1}{5}y^{-4/5}$

23 $\frac{dy}{dx} = 3x^2; \frac{dx}{dy} = \frac{1}{3}(1+y)^{-2/3}$ **25** $\frac{dy}{dx} = \frac{-1}{(x-1)^2}, \frac{dx}{dy} = \frac{-1}{(y-1)^2}$ **27** $y; \frac{1}{2}y^2 + C$

29 $f(g(x)) = -1/3x^3; g^{-1}(y) = \frac{-1}{y}; g(g^{-1}(x)) = x$ **39** $2/\sqrt{3}$ **41** $1/6\cos 9$