25 -1 **27** 1; $\frac{1-\sin x}{1+\cos x}$ has no limit **29** $f'(c) = \frac{4^3-1^3}{4-1}; c = \sqrt{7}$

31 $0 = x^* - x_{n+1} + \frac{f''(c)}{2f'(x_n)}(x^* - x_n)^2$ gives $M \approx \frac{f''(x^*)}{2f'(x^*)}$ **33** $f'(0)$; $\frac{f'(x)}{1}$; singularity **35** $\frac{f(x)}{g(x)} \to \frac{3}{4}$ **37** 1

2 $\sin 2\pi - \sin 0 = (\pi \cos \pi c)(2-0)$ when $\cos \pi c = 0$: then $\mathbf{c = \frac{1}{2}}$ or $\mathbf{c = \frac{3}{2}}$.

4 $(1+2+4) - (1+0+0) = (1+2c)(2-0)$ when $6 = 2(1+2c) =$ or $\mathbf{c = 1}$. (For parabolas c is always halfway between a and b).

6 $(2-1)^9 - (0-1)^9 = 9(c-1)^8(2-0)$ gives $9(c-1)^8 = 1$: then $\mathbf{c = 1 + (\frac{1}{9})^{1/8}}$ or $\mathbf{c = 1 - (\frac{1}{9})^{1/8}}$.

8 $f(x) =$ step function has $f(1) = 1$ and $f(-1) = 0$. Then $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1}{2}$ but no point c has $f'(c) = \frac{1}{2}$. MVT does not apply because f is **not continuous** in this interval.

10 $f(x) = \frac{1}{x^2}$ has $f(1) = 1$ and $f(-1) = 1$, but no point c has $f'(c) = 0$. MVT does not apply because $f(x)$ is **not continuous** in this interval.

12 $\frac{d}{dx}\csc^2 x = 2\csc x(-\csc x \cot x)$ is equal to $\frac{d}{dx}\cot^2 x = 2\cot x(-\csc^2 x)$. Then $f(x) = \csc^2 x - \cot^2 x$ has $f' = 0$ at every point c. By the MVT $f(x)$ must have the same value at every pair of points a and b. By trigonometry $\csc^2 x - \cos^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1$ at all points.

14 $\lim_{x\to 3} \frac{x^2-9}{x+3} = \frac{0}{6}$. (This is not a case for l'Hôpital's Rule! It is just limit of $f(x)$ divided by limit of $g(x)$.)

16 $L = \lim_{x\to 0} \frac{\sqrt{1-\cos x}}{x} =$ (by l'Hôpital's Rule) $\lim_{x\to 0} \frac{\frac{1}{2}(1-\cos x)^{-1/2}\sin x}{1}$ or $L = \lim \frac{\sin x}{2\sqrt{1-\cos x}}$. This is again $\frac{0}{0}$. But we can multiply by $\frac{x}{\sin x} \to 1$ to reach $\lim_{x\to 0} \frac{x}{2\sqrt{1-\cos x}}$ which is $\frac{1}{2L}$. Thus $L = \frac{1}{2L}$ and $\mathbf{L = \frac{1}{\sqrt{2}}}$. (Note: The knowledge that $1 - \cos x \approx \frac{x^2}{2}$ also gives $\lim \frac{\sqrt{1-\cos x}}{x} = \lim \frac{x/\sqrt{2}}{x} = \frac{1}{\sqrt{2}}$.)

18 $\lim_{x\to 1} \frac{x-1}{\sin x} = \frac{0}{\sin 1} = \mathbf{0}$ (not an application of l'Hôpital's Rule).

20 $\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x^2} = \lim_{x\to 0} \frac{n(1+x)^{n-1} - n}{2x} =$ (l'Hôpital again) $\lim_{x\to 0} \frac{n(n-1)(1+x)^{n-2}}{2} = \mathbf{\frac{n(n-1)}{2}}$.

22 $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}}{1} = \frac{1}{2} + \frac{1}{2} = \mathbf{1}$.

24 The steps when $f \to \infty$ and $g \to \infty$ are $L = \lim \frac{f}{g} = \lim \frac{1/g}{1/f} =$ **(now comes l'Hôpital for $\frac{0}{0}$)** $\lim \frac{g'/g^2}{f'/f^2} =$ **(here is the limit of a product or quotient)** $(\lim \frac{f^2}{g^2})/\lim \frac{f'}{g'} = L^2/\lim \frac{f'}{g'}$. Cancel L to find $\lim \frac{f'}{g'} = L$.

26 $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$ approaches $\frac{\infty}{\infty}$. ok to use l'Hôpital: find $\lim_{x\to 0} \frac{-1/x^2}{1/x^2} = \mathbf{-1}$. Also ok to rewrite the original ratio: $\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{x+1}{x-1}$ which approaches $\frac{+1}{-1} = \mathbf{-1}$.

28 $\frac{\csc x}{\cot x} = \frac{1}{\sin x}\frac{\sin x}{\cos x} = \frac{1}{\cos x}$. The limit as $x \to 0$ is $\frac{1}{1} = \mathbf{1}$. ok to use l'Hôpital's Rule for $\frac{\infty}{\infty}$: $L = \lim_{x\to 0} \frac{\csc x}{\cot x} = \lim_{x\to 0} \frac{-\csc x \cot x}{-\csc^2 x} = \lim \frac{\cot x}{\csc x} = \frac{1}{L}$. This gives $L^2 = 1$ but does not eliminate $L = -1$; add the fact that $\csc x$ and $\cot x$ have the same sign near $x = 0$.

30 Mean Value Theorem: $f(x) - f(y) = f'(c)(x-y)$. Therefore $|f(x) - f(y)| = |f'(c)||x-y| \le |x-y|$ since we are given that $|f'| \le 1$ at all points. Geometric interpretation: If the tangent slope stays between -1 and 1, so does the slope of any secant line.

32 **No:** The converse of Rolle's theorem is false. The function $f(x) = x^3$ has $f' = 0$ at $x = 0$ (horizontal tangent). But there are no two points where $f(a) = f(b)$ (no horizontal secant line).

34 $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{x} = \lim_{x\to 0}(x\cos\frac{1}{x}) = 0$ because always $|x\cos\frac{1}{x}| \le |x|$. However $\frac{f'(x)}{g'(x)} = \frac{\sin\frac{1}{x} + 2x\cos\frac{1}{x}}{1}$ has no limit because $\sin\frac{1}{x}$ oscillates as $x \to 0$ (its graph is in Section 2.7).

36 If you travel 3000 miles in 100 hours then at some moment your speed is **30 miles per hour.**

38 Mean Value Theorem: $f(b) - f(a) = f'(c)(b-a)$ is positive if $f'(c) > 0$ and $b > a$. Therefore $f(b) - f(a) > 0$ and $f(b) > f(a)$. A function with positive slope is increasing (as stated without proof in Section 3.2).