the shift $X = x - 2$ and $Y = y + 12$ centers the equation to $Y = 3X^2$.

**4** $y = (x-1)^2$ has vertex at **(1,0)** and focus above it at $(1, \frac{1}{4})$. Note $a = 1$.

**6** $4x = y^2$ has vertex at (0,0) and opens to the right. The focus is at $(\frac{1}{16}, 0)$. (Note the coefficient of $y^2$ in $x = \frac{1}{4}y^2$.)

**8** $x^2 + 9y^2 = 9$ is the ellipse $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$ with vertices $(\pm 3, 0)$ on the major axis and foci $(\pm\sqrt{8}, 0)$.

**10** $\frac{x^2}{2^2} - \frac{(y-1)^2}{1^2} = 1$ is a hyperbola centered at (0,1). It opens right and left with vertices at $(\pm 2, 1)$ and foci at $(\pm\sqrt{5}, 1)$.

**12** $(y-1)^2 - \frac{x^2}{(1/2)^2} = 1$ is a hyperbola centered at (0,1): $a = 1$ and $b = \frac{1}{2}$. It opens up and down with vertices at (0,2) and (0,0). The foci are $(0, 1 \pm \sqrt{1 + \frac{1}{4}})$.

**14** $xy = 0$ gives the two lines $x = 0$ and $y = 0$, a degenerate hyperbola with vertices and foci all at (0,0).

**16** $y = x^2 - x$ has vertex at $(\frac{1}{2}, -\frac{1}{4})$. To move the vertex to (0,0) set $X = x - \frac{1}{2}$ and $Y = y + \frac{1}{4}$. Then $Y = X^2$.

**18** The parabola $y = 9 - x^2$ opens down with vertex at (0,9).

**20** The path $x = t,\ y = t - t^2$ starts with $\frac{dx}{dt} = \frac{dy}{dt} = 1$ at $t = 0$ (45° angle). Then $y_{\max} = \frac{1}{4}$ at $t = \frac{1}{2}$. The path is the parabola $y = x - x^2$.

**22** When $x^2 = a^2 - b^2$ the equation gives $\frac{a^2-b^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} = \frac{b^2}{a^2}$ or $\mathbf{y = \frac{b^2}{a}}$. This is the height above the focus.

**24** Squaring $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$ yields $x^2 - 2cx + c^2 + y^2 = 4a^2 + x^2 + 2cx + c^2 + y^2$ $-4a\sqrt{(x+c)^2 + y^2}$. This is $-4cx - 4a^2 = -4a\sqrt{(x+c)^2 + y^2}$. Divide by $-4$ and square again: $c^2x^2 + 2a^2cx + a^4 = a^2(x^2 + 2cx + c^2 + y^2)$. Set $c^2 = a^2 - b^2$ and cancel $a^2x^2 + 2a^2cx + a^4$ to leave $-b^2x^2 = -a^2b^2 + a^2y^2$. Divide by $a^2b^2$ to find $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the ellipse!).

**26** At $z = 0$ the equation becomes $(x - 2y)^2 + (y - 2x)^2 = 1$ or $\mathbf{5x^2 - 8xy + 5y^2 = 1}$. Then $B^2 = \mathbf{64} < 4AC = \mathbf{100}$ (ellipse).

**28** (a) The line touches the circle at $(x_0, y_0)$ because $x_0^2 + y_0^2 = r^2$. It is tangent because the slope $\frac{dy}{dx} = -\mathbf{\frac{x_0}{y_0}}$ is perpendicular to the slope $\frac{y_0}{x_0}$ of the radius (product of slopes is 1). (b) The derivative of the ellipse equation is $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\mathbf{\frac{x_0\, b^2}{y_0\, a^2}}$ = slope of the line. (This is **implicit differentiation.**)

**30** $PF_1 = PR$ and $QF_1 = QR$ (because $F_1$ and $R$ are mirror images) so step 2 is the same as step 1. Since step 2 holds for every $Q$ on the tangent line, $P$ must be on the straight line from $F_2$ to $R$. The intersection angles of these lines are $\alpha = \beta$.

**32** The square has side $s$ if the point $(\frac{s}{2}, \frac{s}{2})$ is on the ellipse. This requires $\frac{1}{a^2}(\frac{s}{2})^2 + \frac{1}{b^2}(\frac{s}{2})^2 = 1$ or $s^2 = 4(\frac{1}{a^2} + \frac{1}{b^2})^{-1}$ = area of square.

**34** The Earth has $a = 149{,}597{,}870$ kilometers (Problem 19 on page 469 says $1.5 \cdot 10^8$ km). The eccentricity $e = \frac{c}{a}$ is 0.167 (or .02 on page 356). Then $c = 2.5 \cdot 10^6$ and $b = \sqrt{a^2 - c^2}$. This $b$ is very near $a$; our orbit is nearly a circle. Use $\sqrt{a^2 - c^2} \approx a - \frac{c^2}{2a} \approx a - 2 \cdot 10^4$ km.

**36** The derivative of $y^2 - x^2 = 1$ is $2y\frac{dy}{dx} - 2x = 0$ (again implicit). Then $\frac{dy}{dx} = \mathbf{\frac{x_0}{y_0}}$ at the point $(x_0, y_0)$ which agrees with the slope of the line. The line goes through the point because $y_0^2 - x_0^2 = 1$.

**38** The cannon was on a **hyperbola** with foci at **Napoleon** and the **Duke of Wellington**. The hyperbola has $2a$ = distance traveled by sound in 1 second.

**40** Complete squares: $y^2 + 2y = (y+1)^2 - 1$ and $x^2 + 10x = (x+5)^2 - 25$. Then $Y = y + 1$ and $X = x + 5$ satisfy $Y^2 - 1 = X^2 - 25$ : the hyperbola is $\mathbf{X^2 - Y^2 = 24}$.

**42** The graph is empty if $A, C, F$ have the **same sign.**

**44** Given any **five** points in the plane, a second-degree curve goes through those points.

**46** The quadratic $ax^2 + bx + c$ has two real roots if $b^2 - 4ac$ is positive and no real roots if $b^2 - 4ac$

is negative. Equal roots if $b^2 = 4ac$.

## 3.6 Iterations $x_{n+1} = F(x_n)$ (page 136)

$x_{n+1} = x_n^3$ describes an **iteration**. After one step $x_1 = \mathbf{x_0^3}$. After two steps $x_2 = F(x_1) = \mathbf{x_1^3} = \mathbf{x_0^9}$. If it happens that input = output, or $x^* = \mathbf{F(x^*)}$, then $x^*$ is a **fixed** point. $F = x^3$ has **three** fixed points, at $x^* = \mathbf{0, 1}$ **and** $-\mathbf{1}$. Starting near a fixed point, the $x_n$ will converge to it if $|\mathbf{F'(x^*)}| < 1$. That is because $x_{n+1} - x^* = F(x_n) - F(x^*) \approx \mathbf{F'(x^*)(x_n - x^*)}$. The point is called **attracting.** The $x_n$ are repelled if $|\mathbf{F'(x^*)}| > \mathbf{1}$. For $F = x^3$ the fixed points have $F' = \mathbf{0}$ **or 3**. The cobweb goes from $(x_0, x_0)$ to $(x_0, x_1)$ to $\mathbf{(x_1, x_1)}$ and converges to $(x^*, x^*) = \mathbf{(0, 0)}$. This is an intersection of $y = x^3$ and $y = \mathbf{x}$, and it is super-attracting because $\mathbf{F'} = \mathbf{0}$.

$f(x) = 0$ can be solved iteratively by $x_{n+1} = x_n - cf(x_n)$, in which case $F'(x^*) = \mathbf{1 - cf'(x^*)}$. Subtracting $x^* = x^* - cf(x^*)$, the error equation is $x_{n+1} - x^* \approx m\mathbf{(x_n - x^*)}$. The multiplier is $m = \mathbf{1 - cf'(x^*)}$. The errors approach zero if $-\mathbf{1} < \mathbf{m} < \mathbf{1}$. The choice $c_n = \mathbf{1/f'(x^*)}$ produces Newton's method. The choice $c = 1$ is "successive **substitution**" and $c = \mathbf{1/f'(x_0)}$ is modified Newton. Convergence to $x^*$ is **not** certain.

We have three ways to study iterations $x_{n+1} = F(x_n)$ : (1) compute $x_1, x_2, \cdots$ from different $x_0$
(2) find the fixed points $x^*$ and test $|dF/dx| < 1$ (3) draw cobwebs.

**1** $-.366; \infty$ **3** $1; 1$ **5** $\frac{2}{3}; \pm\infty$ **7** $-2; -2$
**9** $\frac{1-\sqrt{3}}{2}$ attracts, $\frac{1+\sqrt{3}}{2}$ repels; $\frac{1}{2}$ attracts, 0 repels; 1 attracts, 0 repels; 1 attracts; $\frac{2}{3}$ attracts, 0 repels; $\pm\sqrt{2}$ repel
**11** Negative **13** .900 **15** .679 **17** $|a| < 1$ **19** Unstable $|F'| > 1$ **21** $x^* = \frac{s}{1-a}; |a| < 1$
**23** \$2000; \$2000 **25** $x_0, b/x_0, x_0, b/x_0, \cdots$ **27** $F' = -\frac{\sqrt{2}}{2}x^{-3/2} = -\frac{1}{2}$ at $x^*$
**29** $F' = 1 - 2cx = 1 - 4c$ at $x^* = 2; 0 < c < \frac{1}{2}$ succeeds
**31** $F' = 1 - 9c(x-2)^8 = 1 - 9c$ at $x^* = 3; 0 < c < \frac{2}{9}$ succeeds
**33** $x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2}; x_{n+1} = x_n - \frac{\sin x_n - \frac{1}{2}}{\cos x_n}$ **35** $x^* = 4$ if $x_0 > 2.5; x^* = 1$ if $x_0 < 2.5$
**37** $m = 1 + c$ at $x^* = 0, m = 1 - c$ at $x^* = 1$ (converges if $0 < c < 2$) **39** 0 **43** $F' = 1$ at $x^* = 0$

**2** $x_{n+1} = 2x_n(1 - x_n) : x_0 = .6, x_1 = .48, x_2 = .4992, \cdots$ approaches $x^* = .5$ and $x_0 = 2, x_1 = -4, x_2 = 8, \cdots$ approaches infinity.
**4** $x_{n+1} = x_n^{-1/2} : x_0 = .6, x_1 = 1.29, x_2 = .88, \cdots$ and $x_0 = 2, x_1 = .707, x_2 = 1.19, \cdots$ both approach $x^* = 1$.
**6** $x_{n+1} = x_n^2 + x_n - 2 : x_0 = .6, x_1 = -1.04, x_2 = -1.9584, \cdots$ and $x_0 = 2, x_1 = 4, x_2 = 18, \cdots$ both approach $+\infty$!
**8** $x_0 = .6, x_1 = .6 \cdots$ approaches $x^* = .6$ and $x_0 = 2, x_1 = 2, \cdots$ approaches $x^* = 2$. Every non-negative number is a fixed point $x^*$.
**10** $x_0 = -1, x_1 = 1, x_2 = -1, x_3 = 1, \cdots$ The double step $x_{n+2} = x_n^9$ has fixed points $x^* = (x^*)^9$, which allows $x^* = 1$ and $x^* = -1$.
**12** $x_{n+1} = x_n^2 - 1 : x_0 = 0, x_1 = -1, x_2 = 0, x_3 = -1, \cdots$. For period 2 look at $x_{n+2} = (x_n^2 - 1)^2 - 1$ and solve $x^* = (x^* - 1)^2 - 1$ to find $x^* = 0, -1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ (four period 2 starting points). The sequence $x_0 = .1, x_1 = -.99, -.0199, -.9996, \cdots$ is attracted to $0, -1, 0, -1, \cdots$ (for proof find zero derivative at $x = 0$).
**14** $x = \cos^2 x : x = \mathbf{.6417}$ **16** $x = 2x - 1 : x = 1$ but the iteration $x_{n+1} = 2x_n - 1$ blows up
**18** At $x^* = (a-1)/a$ the derivative $f' = a - 2ax$ equals $f'(x^*) = a - 2(a-1) = 2 - a$. Convergence if $|F'(x^*)| < 1$

or $1 < a < 3$. (For completeness check $a = 1$ : convergence to zero. Also check $a = 3$ : with $x_0 = .66666$ my calculator gives back $x_2 = .66666$. Apparently period 2.)

**20** $x^* = (x^*)^2 - \frac{1}{2}$ gives $x^*_+ = \frac{1+\sqrt{3}}{2}$ and $x^*_- = \frac{1-\sqrt{3}}{2}$. At these fixed points $F' = 2x^*$ equals $1+\sqrt{3}$ (greater than 1 so $x^*_+$ repels) and $F'(x^*_-) = 1 - \sqrt{3}(x^*_-$ attracts). Cobwebs show convergence to $x^*_-$ if $|x_0| < |x^*_+|$, convergence to $x^*_+$ if $|x_0| = |x^*_+|$, divergence to $\infty$ if $|x_0| > |x^*_+|$.

**22** $x_{n+1} = x_n + 4$ adds 4 each time and diverges; $x_{n+1} = -x_n + 4$ oscillates around 2. (Example: $x_0 = 1$, $x_1 = 3, x_2 = 1, \cdots$.) The linear term doesn't pull it in to 2 because $|F'| = 1$ exactly.

**24** The debt $x_n$ at year $n$ leads to debt $x_{n+1} = .95x_n + \$100$ billion. As $n \to \infty$ the steady state is $x^* = .95x^* + \$100$ billion or $.05x^* = \$100$ billion or $\mathbf{x^* = \$2}$ **trillion**. If $x_0 = \$1$ trillion then every $x_n$ equals \$2 trillion $-(.95)^n$ (\$1 trillion).

**26** The fixed points satisfy $x^* = (x^*)^2 + x^* - 3$ or $(x^*)^2 = 3$; thus $x^* = \sqrt{3}$ or $x^* = -\sqrt{3}$. The derivative $2x^* + 1$ equals $2\sqrt{3} + 1$ or $-2\sqrt{3} + 1$; both have $|F'| > 1$. The iterations blow up.

**28** (a) Start with $x_0 > 0$. Then $x_1 = \sin x_0$ is less than $x_0$. The sequence $x_0, \sin x_0, \sin(\sin x_0) \cdots$ decreases to zero (**convergence:** also if $x_0 < 0$.) On the other hand $x_1 = \tan x_0$ is larger than $x_0$. The sequence $x_0, \tan x_0, \tan(\tan x_0), \cdots$ is increasing (slowly repelled from 0). Since $(\tan x)' = \sec^2 x \geq 1$ there is no attractor (**divergence**). (b) $F''$ is $(\sin x)'' = -\sin x$ and $(\tan x)'' = 2\sec^2 x \tan x$.
Theory: When $F''$ changes from $+$ to $-$ as $x$ passes $x_0$, the curve stays closer to the axis than the 45° line (convergence). Otherwise divergence. See Problem 22 for $F'' = 0$.

**30** $f(x) = x^2 - 4x + 3$ equals zero at $x^* = 1$ where $f' = -2$; also $f(x) = 0$ at $x^* = 3$ where $f' = 2$. The iteration $x_{n+1} = x_n - cf(x_n)$ has $F' = 1 + 2c$ at $x^* = 1$ and $F' = 1 - 2c$ at $x^* = 3$. For $-1 < c < 0$ it converges to $x^* = 1$; for $0 < c < 1$ it converges to $x^* = 3$; if $|c| > 1$ it diverges because $|F'| > 1$ at both fixed points.

**32** $f(x) = \frac{1}{1-x} - 3$ equals zero when $1 - x = \frac{1}{3}$ or $x^* = \frac{2}{3}$; at that point $f' = \frac{1}{(1-x)^2} = 9$. The iteration $x_{n+1} = x_n - cf(x_n)$ has $F' = 1 - 9c$ at $x^*$. For $0 < c < \frac{2}{9}$ it converges because then $|F'(x^*)| < 1$.

**34** Newton's method for $f(x) = x^3 - 2 = 0$ is $x_{n+1} = x_n - \frac{1}{3x_n^2}(x_n^3 - 2)$; convergence to $x^* = 2^{1/3} = 1.259921$. Newton's method for $f(x) = \sin x - \frac{1}{2}$ is $x_{n+1} = x_n - \frac{1}{\cos x_n}(\sin x_n - \frac{1}{2})$; convergence (from nearby $x_0$) to $x^* = \frac{\pi}{6} = .523598$.

**36** Newton's method for $f(x) = x^2 - 1 = 0$ is $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 - 1) = \mathbf{\frac{x_n}{2} + \frac{1}{2x_n}}$. If $x_0 = 10^6$ then $x_1 = \frac{1}{2}10^6 + \frac{1}{2}10^{-6}$. The distance from $x^* = 1$ was $10^6 - 1$; it is cut approximately in half. But if $x_0$ is close to 1 the multiplier is near zero: $x_0 = 1.05$ gives $x_1 = \frac{1.05}{2} + \frac{1}{2.1} \approx 1.001$.

**38** The iterations from $x_0 = 1$ are $x_{n+1} = x_n - (x_n^2 - \frac{1}{2})$ and $x_{n+1} = x_n - \frac{1}{2}(x_n^2 - \frac{1}{2})$ and $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 - \frac{1}{2})$. After one step they give $x_1 = \frac{1}{2}$ and $x_1 = \frac{3}{4}$ and $x_1 = \frac{3}{4}$. After two steps $x_1 = \frac{3}{4} = \mathbf{.750}$ and $x_1 = \frac{23}{32} = \mathbf{.719}$ and $x_1 = \frac{17}{24} = \mathbf{.708}$ (with $x^* = .707$).

**40** The roots of $x^2 + 2 = 0$ are imaginary, and Newton's method $x_{n+1} = x_n - \frac{1}{2x_n}(x_n^2 + 2)$ stays real: convergence is impossible. However the $x_n$ do not approach infinity (if $x_n$ is very large, then $x_{n+1}$ is only half as large). Section 3.7 shows how the $x_n$ jump around chaotically.

**42** The graphs of $\cos x, \cos(\cos x), \cos(\cos(\cos x))$ are approaching the **horizontal line** $\mathbf{y = .7391} \cdots$ (where $x^* = \cos x^*$). For every $x$ this number is the limit.

## 3.7 Newton's Method and Chaos (page 145)

When $f(x) = 0$ is linearized to $f(x_n) + f'(x_n)(x - x_n) = 0$, the solution $x = \mathbf{x_n - f(x_n)/f'(x_n)}$ is Newton's $x_{n+1}$. The **tangent line** to the curve crosses the axis at $x_{n+1}$, while the **curve** crosses at $x^*$. The errors at $x_n$ and $x_{n+1}$ are normally related by $(\text{error})_{n+1} \approx M(\textbf{error})^2_{\mathbf{n}}$. This is **quadratic** convergence. The number of correct decimals **doubles** at every step.

For $f(x) = x^2 - b$, Newton's iteration is $x_{n+1} = \frac{1}{2}(b + \frac{x_n}{b})$. The $x_n$ converge to $\sqrt{b}$ if $x_0 > 0$ and to $-\sqrt{b}$ if $x_0 < 0$. For $f(x) = x^2 + 1$, the iteration becomes $x_{n+1} = \frac{1}{2}(x_n - x_n^{-1})$. This cannot converge to $i = \sqrt{-1}$. Instead it leads to chaos. Changing to $z = 1/(x^2+1)$ yields the parabolic iteration $z_{n+1} = 4z_n - 4z_n^2$.

For $a \le 3, z_{n+1} = az_n - az_n^2$ converges to a single **fixed point.** After $a = 3$ the limit is a 2-cycle, which means **that the z's alternate between two values.** Later the limit is a Cantor set, which is a one-dimensional example of a **fractal.** The Cantor set is **self-similar.**

**1** $x_{n+1} = x_n - \frac{x_n^3 - b}{3x_n^2} = \frac{2x_n}{3} + \frac{b}{3x_n^2}$ **5** $x_1 = x_0; x_1$ is not defined $(\infty)$ **7** $x^* = 2$; blows up; $x^* = 2$ if $x_0 < 3$
**11** $x_0 < \frac{1}{2}$ to $x^* = 0; x_0 > \frac{1}{2}$ to $x^* = 1$ **21** $x_{n+1} = x_n - \frac{x_n^k - 7}{kx_n^{k-1}}$ **23** $x_4 = \cot\pi = \infty; x_3 = \cot\frac{8\pi}{7} = \cot\frac{\pi}{7}$
**25** $\pi$ is not a fraction **27** $= \frac{1}{4}x_n^2 + \frac{1}{2} + \frac{1}{4x_n^2} = \frac{(x_n^2+1)^2}{4x_n^2} = \frac{y_n^2}{4(y_n - 1)}$ **29** $16z - 80z^2 + 128z^3 - 64z^4; 4; 2$
**31** $|x_0| < 1$ **33** $\Delta x = 1$, one-step convergence for quadratics **35** $\frac{\Delta f}{\Delta x} = \frac{5.25}{1.5}; x_2 = 1.86$
**37** $1.75 < x^* < 2.5; 1.75 < x^* < 2.125$ **39** $8; 3 < x^* < 4$ **41** Increases by 1; doubles for Newton
**45** $x_1 = x_0 + \cot x_0 = x_0 + \pi$ gives $x_2 = x_1 + \cot x_1 = x_1 + \pi$ **49** $a = 2, Y$'s approach $\frac{1}{2}$

**2** $f(x) = \frac{x-1}{x+1}$ has $f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$ so Newton's formula is $x_{n+1} = x_n - \frac{(x_n+1)^2}{2}\frac{x_n-1}{x_n+1} = x_n - \frac{x_n^2-1}{2}$. The fixed points of this $F$ satisfy $x^* = x^* - \frac{(x^*)^2-1}{2}$ which gives $x^* = 1$ and $x^* = -1$. The derivatives $F' = 1 - x^*$ are 0 and $-2$. So the sequence approaches $x^* = 1$, the correct zero of $f(x)$.

**4** $f(x) = x^{1/3}$ has $f'(x) = \frac{1}{3x^{2/3}}$ so Newton's formula is $x_{n+1} = x_n - 3x_n^{2/3}(x_n^{1/3}) = -2x_n$. The graph of $x^{1/3}$ is vertical at $x = 0$; the tangent line at any $x$ hits the axis at $-2x$.

**6** $f(x) = x^3 - 3x - 1 = 0$: roots near $1.9, -.5, -1.6$

**8** For any $x_0$, the new $x_1$ is on the right side of the root. Then $x_2, x_3, \cdots$ approach steadily from the right.

**10** Newton's method for $f(x) = x^4 - 100$ approaches $x^* = \sqrt{10}$ if $x_0 > 0$ and $x^* = -\sqrt{10}$ if $x_0 < 0$. In this case the error at step $n+1$ equals $\frac{3}{2x^*}$ times (error at step $n)^2$. In Problem 9 the multiplier is $\frac{1}{2x^*}$ and convergence is quicker. Note to instructors: The multiplier is $\frac{f''(x^*)}{2f'(x^*)}$ (this is $\frac{1}{2}F''(x^*)$ : see Problem 31 of Section 3.8).

**12** $x^3 - x = 0$ gives $\mathbf{x^* = 1, 0,}$ **and** $\mathbf{-1}$. Newton's method has $x_1 = x_0 - \frac{x_0^3 - x_0}{3x_0^2 - 1} = \frac{2x_0^3}{3x_0^2-1}$. This equals $-x_0$ (producing a cycle) if $x_0 = \pm\sqrt{.2}$. Between these limits we have $|x_1| < |x_0|$ and Newton converges to $x^* = 0$. Between $|x_0| = \sqrt{.2}$ and $|x_0| = \sqrt{1/3}$ the convergence to 1 or $-1$ looks complicated. For $x_0 > \sqrt{1/3}$ there is convergence to $x^* = 1$.

**14** Between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ the graph of $-\tan x$ decreases from $\infty$ to $-\infty$. It crosses the 45° line once (at $x = 0$). In each successive interval of length $\pi$, the same is true: one solution to $x = -\tan x$ in each interval. Roots $\mathbf{x^* = 0}$ and $x^* = 2.03$.

**16** Roots at $x^* = -1.3$ and $x^* = .526$.

**18** (a) From $1 - 2x_{n+1} = (1 - 2x_n)^2 = 1 - 4x_n + 4x_n^2$, cancel the 1's and divide by $-2$. Then $x_{n+1} = 2x_n - 2x_n^2$.
(b) Every step squares $1 - 2x_n$ to find the next $1 - 2x_{n+1}$. So if $|1 - 2x_0| > 1$, repeated squaring blows up. If $|1 - 2x_0| < 1$, then repeated squaring gives $1 - 2x_n \to 0$. Here $|1 - 2x_0| < 1$ puts $2x_0$ between 0 and 2.

**20** Multiply $x_{n+1} = 2x_n - ax_n^2$ by $a$ and subtract from 1 to find $1 - ax_{n+1} = 1 - 2ax_n + a^2x_n^2 = (1 - ax_n)^2$. At each step $1 - ax_n$ is *squared* to find the next $1 - ax_{n+1}$. Then $1 - ax_n \to 0$ (or $x_n \to \frac{1}{a}$) if $|1 - ax_0| < 1$. This puts $1 - ax_0$ between $-1$ and 1: then $0 < x_0 < 2/a$.

**22** Roots at $x^* = -2.11485$ and $x^* = .25410$ and $x^* = 1.86081$.

**24** $\theta = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}, \frac{32\pi}{9}, \frac{64\pi}{9} = 7\pi + \frac{\pi}{9}$; this happened at step 6 so $x_6 = x_0$.

**26** If $z_0 = \sin^2\theta$ then $z_1 = 4z_0 - 4z_0^2 = 4\sin^2\theta - 4\sin^4\theta = 4\sin^2\theta(1-\sin^2\theta) = 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2 = \sin^2 2\theta$.

**28** $\frac{1}{y_{n+1}} = \frac{4(y_n-1)}{y_n^2} = \frac{4}{y_n} - \frac{4}{y_n^2}$. With $z = \frac{1}{y}$ this is $\mathbf{z_{n+1} = 4z_n - 4z_n^2}$.

**30** Newton's method is $x_{n+1} = x_n - \frac{3x_n^2 - .64}{x_n^3 - .64x_n - .36}$. TO DO

**32** The function $f(x) = x^2$ has a *double root* at $x^* = 0$. Newton's iteration is $x_{n+1} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}$. Each step multiplies by $\frac{1}{2}$ : there is convergence to $x^* = 0$ but it is only *linear*. The error is not squared.

**34** Halley's method to solve $f(x) = x^2 - 1 = 0$ is $(x_n^2 - 1) + \Delta x(2x_n) + \frac{\Delta x}{2}\frac{1-x_n^2}{2x_n}(2) = 0$ or $\Delta x = \frac{1-x_n^2}{2x_n + \frac{1-x_n^2}{2x_n}}$. (This is Newton's method with an extra term in the denominator.) Substitute $x_0 = 1+\epsilon$ to find $\Delta x = \frac{-2\epsilon-\epsilon^2}{2+2\epsilon+\frac{-2\epsilon-\epsilon^2}{2+2\epsilon}}$. After some calculation $x_1 = x_0 + \Delta x = 1 + \epsilon + \Delta x$ is $1 + O(\epsilon^3)$.

**36** The secant line connecting $x_0 = 1, f(x_0) = -3$ to the next point $x_2 = 2.5, f(x_2) = 2.25$ has slope $\frac{\Delta f}{\Delta x} = \frac{5.25}{1.5}$. The line with this slope is $y + 3 = \frac{5.25}{1.5}(x-1)$. It crosses $y = 0$ at the point $x_2 = 1 + 3\frac{1.5}{5.25} = 1.857$.

**40** Root at $x^* = .29$ (and very flat nearby)

**42** $\frac{3}{4} = \frac{2}{3} + \frac{2}{27} + \frac{2}{9(27)} + \cdots$ (This is .2020 $\cdots$ not in decimals but when the base is changed to 3.) The Cantor set removes the interval $(\frac{1}{3}, \frac{2}{3})$ where the first digit is 1; then $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ where the second digit is 1; eventually all numbers containing a "1" in the expansion to base 3 are removed. The number $\frac{3}{4} = .202020\cdots$ is not removed – it remains in the Cantor set.

**44** A Newton step goes from $x_0 = .308$ to $x_1 = x_0 + \frac{\cos x_0}{\sin x_0} = 3.45143$. Then $\frac{\Delta f}{\Delta x} = \frac{\cos x_1 - \cos x_0}{x_1 - x_0} = -.606129$ and a secant step leads to $x_2 = x_1 + \frac{\cos x_1}{.606129} = 1.88$.

**48** The graphs of $Y_1(Y_1(Y_1\cdots(x)))$ become squarer and squarer, going between heights .842 and .452. $Y_9$ is like $Y_8$ but "flipped" – because $Y_1(.842) = .452$ and $Y_1(.452) = .842$. These are fixed points of $Y_1(Y_1(x))$– draw its intersection with the 45° line $y = x$. Note that $Y_9$ is a polynomial of degree $2^9$. *Unusual graphs!*

## 3.8 The Mean Value Theorem and l'Hôpital's Rule (page 152)

The Mean Value Theorem equates the average slope $\Delta f/\Delta x$ over an **interval** $[a, b]$ to the slope $df/dx$ at an unknown **point**. The statement is $\mathbf{\Delta f/\Delta x = f'(x)}$ **for some point a < c < b**. It requires $f(x)$ to be **continuous** on the **closed** interval $[a, b]$, with a **derivative** on the open interval $(a, b)$. Rolle's theorem is the special case when $f(a) = f(b) = 0$, and the point $c$ satisfies $\mathbf{f'(c) = 0}$. The proof chooses $c$ as the point where $f$ reaches its **maximum or minimum**.

Consequences of the Mean Value Theorem include: If $f'(x) = 0$ everywhere in an interval then $f(x) =$ **constant**. The prediction $f(x) = f(a) + \mathbf{f'(c)}(x-a)$ is exact for some $c$ between $a$ and $x$. The quadratic prediction $f(x) = f(a) + f'(x)(x-a) + \frac{1}{2}\mathbf{f''(c)}(x-a)^2$ is exact for another $c$. The error in $f(a) + f'(a)(x-a)$ is less than $\frac{1}{2}M(x-a)^2$ where $M$ is the maximum of $|\mathbf{f''}|$.

A chief consequence is l'Hôpital's Rule, which applies when $f(x)$ and $g(x) \to \mathbf{0}$ as $x \to a$. In that case the limit of $f(x)/g(x)$ equals the limit of $\mathbf{f'(x)/g'(x)}$, provided this limit exists. Normally this limit is $f'(a)/g'(a)$. If this is also 0/0, go on to the limit of $\mathbf{f''(x)/g''(x)}$.

**1** $c = \sqrt{\frac{4}{3}}$ **3** No $c$ **5** $c = 1$ **7** Corner at $\frac{1}{2}$ **9** Cusp at 0

**11** $\sec^2 x - \tan^2 x =$ constant **13** 6 **15** $-2$ **17** $-1$ **19** $n$ **21** $-\frac{1}{2}$ **23** Not $\frac{0}{0}$