Calculus Week 8 solutions, odd

3.8 : 1, 7, 13, 17, 29   4.1 : 7, 13, 19, 29, 31, 37, 51   4.2 : 5, 7, 11, 21, 23

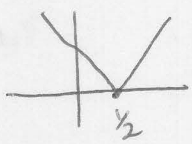
3.8 #1 $f(x)=x^3$

$f'(x)=3x^2$

$f'(c)=3c^2 = \frac{f(2)-f(0)}{2-0} = \frac{8-0}{2} = 4 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$ (but only + is in $0<x<2$)

$C = \frac{+2}{\sqrt{3}}$

7) $f(x)=|x-\frac{1}{2}|$ $f'(c) = \frac{f(1)-f(-1)}{1+1} = \frac{\frac{1}{2}-\frac{3}{2}}{2} = -\frac{1}{2}$ but slopes only -1 and 1.



MVT doesn't apply since f is not differentiable at $x=\frac{1}{2}$

13) $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = \lim\limits_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim\limits_{x\to 3} x+3 = 6$. Check w/ L'Hôpital: $\lim\limits_{x\to 3} \frac{2x}{1} = 2\cdot 3 = 6$ ✓

17) $\lim\limits_{x\to\pi} \frac{x-\pi}{\sin x} = \lim\limits_{x\to\pi} \frac{1}{\cos\pi} = -1$

29) MVT guarantees that some c between 1 and 4 satisfies $f'(c) = 3c^2 = \frac{4^3-1^3}{4-1} = \frac{63}{3} = 21$

so $c^2 = 7$ and $c=\sqrt{7}$

4.1

#7 $z = \tan(\frac{1}{x}) + \frac{1}{\tan x}$ $f(x)=\tan x$ $g(x)=\frac{1}{x}$ $z' = \frac{-1}{x^2}\sec^2(\frac{1}{x}) - \frac{1}{\tan^2 x}\cdot\sec^2 x$

$= f(g(x)) + g(f(x))$

#13 $\cos(\cos(x))$ $\frac{dz}{dx} = -\sin(\cos x)\cdot(-\sin x) = \sin x\cdot\sin(\cos x)$

#19 $z=\sqrt{1+\sin x}$ $\frac{dz}{dx} = \frac{1}{2}(1+\sin x)^{-1/2}\cdot\cos x = \frac{\cos x}{2\sqrt{1+\sin x}}$

29) $f(y)=y-2$

$f(g(x)) = g(x)-2 = x \Rightarrow g(x) = x+2$

$f(h(x)) = h(x)-2 = x^2 \Rightarrow h(x) = x^2+2$

$f(k(x)) = k(x)-2 = 1 \Rightarrow k(x) = 3$

31) $\frac{d}{dx} f(f(x)) = f'(f(x))\cdot f'(x) \neq [f'(x)]^2 = \left(\frac{df}{dx}\right)^2$. So, not equal.

If $f(x) = \frac{1}{x}$, $f'(f(x))\cdot f'(x) = \frac{-1}{(\frac{1}{x})^2}\cdot(\frac{-1}{x^2}) = 1$ and $f(f(x)) = \frac{1}{1/x} = x$ ✓.

37) $g(x) = 1-x \qquad g(f(x)) = 1 - \frac{1}{x} = \frac{x-1}{x} \qquad f(g(f(x))) = \frac{x}{x-1}$

$f(x) = \frac{1}{x} \qquad f(g(x)) = \frac{1}{1-x} \qquad g(f(g(x))) = 1 - \frac{1}{1-x} = \frac{1-x-1}{1-x} = \frac{x}{x-1}$

Furthermore, $f(f(x)) = x \quad g(g(x)) = 1-(1-x) = x$

After this, we just get these 6 functions back again, with further compositions.

$f(f(f(f(x)))) = \frac{1}{x}, \quad f(f(g(f(x)))) = \frac{x-1}{x}, \quad g(f(g(f(x)))) = 1 - \frac{x}{x-1} = \frac{1}{1-x}$ etc.

51) $Z = \sqrt{1-u} \qquad u = \sqrt{1-x} \qquad \frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx} = \frac{1}{2}(1-u)^{-1/2} \cdot \frac{1}{2}(1-x)^{-1/2}\cdot(-1) = \cancel{4}$

$= \frac{1}{2}\left(1-\sqrt{1-x}\right)^{-1/2}\cdot\frac{1}{2}(1-x)^{-1/2}(-1)$

4.2: 5, 7, 11, 21, 23

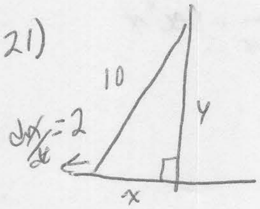
5) $x = F(y) \Rightarrow 1 = F'(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{F'(y)}$

7) $x^2 y = y^2 x \Rightarrow 2xy + x^2\frac{dy}{dx} = 2y\frac{dy}{dx}\cdot x + y^2$

$\Rightarrow \frac{dy}{dx}(x^2 - 2xy) = y^2 - 2xy \Rightarrow \frac{dy}{dx} = \frac{y^2-2xy}{x^2-2xy}$

11) $xy = C \Rightarrow y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$ negative reciprocal slopes $\Rightarrow$ perpendicular

$x^2 - y^2 = D \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

21) 

$x^2 + y^2 = 100$

$\Rightarrow 2x\cdot\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

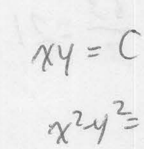
(a) when $y = 6,\ x = 8 \Rightarrow 8\cdot 2 + 6\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-16}{6}$

(b) $y = 5 \Rightarrow x = 5\sqrt{3} \Rightarrow 5\sqrt{3}\cdot 2 + 5\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -2\sqrt{3}$

(c) when $y = 0, x = 10 \Rightarrow$ $\cancel{10\cdot\frac{dy}{dt} = 0}$ $10\cdot 2 + 0\frac{dy}{dt} = 0$

This is impossible, but note $\lim_{y\to 0}\frac{dy}{dt} = \infty$, and then $\frac{dy}{dt} = 0$ when ladder is at rest.

(Of course, the ladder velocity can't really $\to\infty$... there must be more physical conditions we need to include to model reality!)

23) 

$V = \pi r^2 h = 4\pi h$

$\Rightarrow \frac{dV}{dt} = 4\pi\frac{dh}{dt}$ when $\frac{dV}{dt} = -1,\ \frac{dh}{dt} = \frac{-1}{4\pi}$