RELATIVITY

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MATTER and MOTION

SPRING2018

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Selected Readings Taken From
Classical and Modern Physics

Modified by K. Chowdary, The Evergreen State College, 2008, 2011, 2013, 2015, 2018 for the Evergreen academic programs: Knowing Nature/The Nature of Space and Time; Stages of Discovery: Revolutions in Art and Science; Trajectories in Animation, Mathematics, and Physics; Models of Motion; Matter and Motion

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Relativity 1: Basic Postulates

Learning Goals

1. **State the basic, fundamental principles of relativity, and explain how the various aspects of relativity all follow logically from these basic principles.**
2. **Develop your sense for distances and speeds, connecting numerical values to real-world quantities. Convert between standard and relativistic units for distance and for speed.**
3. **Relate time intervals in two different reference frames using the proper time relation if one of the observers is at both events.**
4. **Relate length and distance measurements in two different reference frames using the length contraction relation if one of the observers is at rest with respect to the distance/length being measured.**
5. **Use the velocity transformations to relate the velocities of objects or of reference frames.**

In 2015, we celebrated the 110th anniversary of Albert Einstein's so called *Annus mirabilis*, or "Miracle Year." In 1905, Einstein published four papers that revolutionized science and fundamentally changed the way in which we view the universe. Interestingly, three of these seminal papers were published in the very same issue of the journal *Annalen der Physik*, volume 17. The first paper (beginning on p. 132 of that issue) explained a phenomenon known as the photoelectric effect by introducing the idea of photons (particles of light), an idea which formed one of the cornerstones of quantum mechanics. Though any one of these papers would have been a monumental lifetime achievement for any physicist, Einstein won the 1921 Nobel Prize in physics "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect." The second paper (p. 549) was the first to connect molecular diffusion (spreading of an impurity in a motionless fluid) with random Brownian motion of the individual impurity molecules. The third paper (p. 891) had an innocuous title: "Zur Elektrodynamik bewegter Korpër" ("On the electrodynamics of moving bodies").

But there is nothing even remotely innocuous about the implications of the theory – now known as *Einstein's Special Theory of Relativity* ("special relativity" for short) – presented in that paper. Einstein's theory completely changed common conceptions of space and time, establishing that they are closely interrelated, as we will see. This theory also changed our conceptions of mass and energy, establishing their intimate relationship; this equivalence was elaborated in Einstein's fourth paper of the Miracle Year, in *Annalen der Physik*, **18**, 639. The theory led to an explanation of how stars generate light – the fundamental source of energy in the universe without which life on this planet would not be possible – and (indirectly) led to the development of nuclear energy and weapons. The theory also holds the key to the future development of certain non-fossil fuel energy sources. Studying Einstein's theory of relativity allows you to change your understanding of nature in a fundamental way.

This chapter discusses some of the main ideas and implications of the Special Theory of Relativity, which applies to the motion of objects in *inertial* (constant velocity) reference frames. Einstein's General Theory of Relativity ("general relativity" for short) expands the theory to account for the effects of acceleration and gravitational fields.

A. Preliminaries

A few definitions will be useful.

An ***event*** is something that happens at a particular location at a particular time. It is important to be clear about this, because relativity deals with how different observers measure distances and times between events. For instance, let's say that the penguin on top of Arlo's television set explodes at 7:12 a.m. on a Saturday morning. Rebecca then runs 5 km to a large tower where Rachel receives (at 7:46 a.m.) a free "Math. Math! MATH!" T-shirt from a mutated lab mouse. You could identify two events – the explosion of the penguin and the receipt of the T-shirt – and say that these events are separated by 5 km in space and 44 min in time. Relativity addresses the question of how a *different* observer measures the distance and time between the same two events. (Spoiler! not everyone will agree about the distance and time between events.)

A ***reference frame*** can be thought of as a set of common observers subject to the same conditions. Specifically, we will say that several observers are in the same reference frame if they agree about the distance and time between any two events.

A particularly important kind of reference frame is an ***inertial reference frame***. Observers in an inertial reference frame experience no significant acceleration, nor can they discern any gravitational effects. In an ideal inertial reference frame, the observer would be floating free (hence the name "free float" that is sometimes used to discuss an inertial reference frame), because any non-floating motion would necessarily imply either acceleration or gravitational effects. To analyze behavior in the vicinity of very strong gravitational fields, it is necessary to use general relativity.

Technically, an observer is not in a true inertial reference frame if she is standing on the surface of a planet since there is gravitation. However, there are plenty of situations where non-inertial effects are small enough as to be negligible. In fact, the gravitation from a typical planet is small enough so that the non-inertial effects are negligible, and Special Relativity works perfectly well. So, for example, we will often treat observers moving on a constant velocity train as though they are in an inertial reference frame, even though there is a small gravitational effect.

When dealing with velocities, we have to be careful. A velocity technically has meaning only if there is a reference. So, for example, if Mina is in a car and she is traveling 65 mph toward the west, she is really traveling 65 mph ***relative to the surface of the Earth***. In fact, almost any velocity that people quote in everyday usage is defined relative to the Earth.

Think About This #1.1:
How fast is Mina *really* going if she is in the car in the previous paragraph?

Certainly, anyone who is willing to accept a non-geocentric view of the universe realizes that there is nothing inherently special about the Earth as a reference frame. But scientists have long wondered if there ***is*** some preferred universal reference frame from which all velocities should be defined, some standard by which we could define ***absolute velocities*** for every object in the universe.

In relativity, we will use ***relative velocities***, i.e., velocities will always be defined relative to some reference frame. In fact, one result of relativity is the realization that this is the best way to define velocity. There is no need to choose any special reference frame for the universe; all the results of relative work perfectly well with velocities measured relative to any reference frame that you might choose.

The following statement applies to relative velocities: if observer A measures observer B to be moving at a (relative) speed v in a particular direction, then B measures A to be moving at a (relative) velocity of the same speed v but in the opposite direction.

B. Fundamental Principles of Relativity

Einstein's Special Theory of Relativity is based on a very simple premise, namely

> **The Principle of Relativity**: the laws of physics are the same for observers in different inertial reference frames.

Let's say, for example, that Sina sets up a lab in the first floor of Lab II while Paul sets up an identical lab inside a truck that is driving on Evergreen Parkway with a constant velocity. Whatever physics equations (including fundamental constants) Sina uses to predict and describe the behavior in Sina's lab should work equally well for Paul in Paul's lab.

Not only is this an intuitively reasonable statement, but the argument can be made that the whole field of physics would be useless if this statement weren't true (along with chemistry, biology and engineering as well). After all, what is the point of formulating a set of laws to describe the universe if they only apply to certain observers moving in a certain way?

The question then boils down to this: what *are* the fundamental "laws of physics" that are the same for all observers? At the beginning of the 20th Century, two of the main cornerstones of physics were Newton's Laws of Classical Mechanics and Maxwell's Equations describing electrical and magnetic fields. You were introduced to Newton's Laws and Maxwell's equations earlier this year.

During the 19th Century, there was a tremendous surge of research to describe electric and magnetic phenomena, culminating in the integration of electromagnetic theory into a set of four fundamental laws by James Clerk Maxwell in the late 1800s. Maxwell's results not only unified electricity and magnetism into a single, consistent theory, but also showed for the first time that light is an electromagnetic wave. The theory also showed how to produce a wide variety of different types of electromagnetic waves (light, radio, microwaves, gamma rays, etc.), a prescription that had been successfully tested during the period between Maxwell's theory and Einstein's work on relativity. Maxwell's equations were considered (and still are) by the scientific community to be one of the cornerstones of physical law.

But there was a problem: by the end of the 19th Century, some theorists – most notably Hendrik Lorentz and George Fitzgerald – attempted to generalize Maxwell's equations to apply to observers in any reference frame and found that this could not be done within the framework of Newtonian Classical Mechanics. There arose, effectively, a conflict between the two most widely-accepted cornerstones of physics.

Here is where Einstein came into the picture. Whereas few people had previously had any doubts about the validity of Newtonian Mechanics, Einstein started from the assumption that Maxwell's Equations of electricity and magnetism were a fundamental law of physics that were valid in any reference frame, and then set about re-writing Newton's Laws (generalizing them, actually) to assure that Maxwell's Equations would be valid in any reference frame (hence the title of Einstein's third paper in 1905.)

The argument goes like this: If Maxwell's Equations are valid for observers in any inertial reference frame, then not only the *form* of the equation but also all the constants should be valid in any reference frame. Two of the constants in particular – the *permittivity of free space* ε_0, and the *permeability of free space* μ_0 – when combined together give a value $3.0\text{x}10^8$ m/s, which is the speed of light when it propagates through a vacuum. Based on the fundamental Principle of Relativity (above), the conclusion is staggering: if Maxwell's equations formulate a fundamental law of physics, then the Relativity Principle implies the following consequence:

> **The invariance of the speed of light:** The speed of light in a vacuum c is measured to be $3.0\text{x}10^8$ m/s *by any observer* in any inertial reference frame.

Though this statement runs counter to our intuition, based on common experience, it has been verified experimentally – for example, in 1895, an experiment by Michelson and Morley indicated the invariance of the speed of light in vacuum.

Consider the following sample problems:

Example 1.1. Classical calculation of relative velocities.
(a) Piper is running directly towards Kai'l with a constant speed of 5 m/s with respect to Kai'l. While still running, Piper throws a ball directly towards Kai'l. If the speed of the ball is 15 m/s relative to Piper, how fast is the dart moving in Kai'l's reference frame?

> **Solution :** The answer is what you would think – simply add the speeds to find that the blow dart travels at a speed of 20 m/s from Kai'l's perspective.

(b) Kelsey picks up her ball and aims it in Sophia's direction. Sophia quickly retreats, running away from Kelsey with a constant speed of 5 m/s. While standing still, Kelsey throws her ball towards Sophia at a constant speed 15 m/s measured from Kelsey's reference frame. How fast is the ball moving in Sophia's reference frame?

> **Solution**: Again, the result is what you would think – simply subtract the speeds to find that the blow dart travels at a speed 10 m/s relative to Sophia.

Example 1.2. **Speeds of light pulses.** Andrew is traveling to Saturn. Approaching the planet at a speed of 2.0x10^8 m/s (relative to the planet), he sends a beacon of light to Jesse, who is stationed on Saturn. This pulse of light leaves Andrew's ship with a speed 3.0x10^8 m/s relative to the ship. How fast is the pulse moving relative to Jesse?

> **Solution:** Classically, you should expect that Jesse would view the pulse as moving with a speed of 5.0x10^8 m/s. But this is wrong. Instead, from Jesse's reference frame, the pulse is moving with a speed of 3.0x10^8 m/s! That's just the way it is with light pulses moving in a vacuum – everyone measures the same speed of 3.0x10^8 m/s, regardless of their motion.

Hopefully, you find the results of Example 1.2 to be strange – there is nothing in our everyday experience that would lead us to expect such a result. But numerous experiments have measured the speed of light from a wide variety of reference frames, and the results always agree with the statement of the invariance of *c*.

That the speed of light (in empty space) does not depend on the speed of its source has been demonstrated so convincingly and the value of the speed measured so accurately that the value is now defined to be *exactly* 299,792,458 m/s. By combining this definition of *c* with the definition of the second (in terms of an atomic clock), we no longer need an independent definition of the meter.

C. Units

When working with relativity, it is convenient to express lengths in terms of distance traveled by light in one unit of time. A "light year" for instance is the distance that light travels in one year. An analogy would be to say that the distance between Olympia and Portland is "two car hours" (i.e., it takes 2 hours to get to Portland in a car driving at highway speeds). In fact, you will often hear people using time directly to express a distance: "Oh, it's 2 hours to Portland from here."

We will abbreviate these units as lt·s, lt·min, lt·yr, … for light-second, light-minute and light-year, respectively. Using these units for distance, we can express speeds in terms of lt·s/s, lt·min/min, lt·yr/yr, etc. Since the speed of light c = 1 lt·s/s = 1 lt·min/min = 1 lt·yr/yr, …, the speed of a particle in these units is simply the speed expressed as a fraction of the speed of light.

Example 1.3. Conversion from light-seconds/second to meters/second. Fiona is traveling at a speed of 0.25 lt·s/s. Find Fiona's speed in units of m/s.

> **Solution:** Use the fact that 1 lt·s/s is equal to about 3.00 x 10^8 m/s. Then convert units as follows:

$$0.25\ \mathrm{lt\cdot s/s} \times \frac{3.00\times 10^{8}\,\mathrm{m/s}}{1\ \mathrm{lt\cdot s/s}} = 0.75\times 10^{8}\,\mathrm{m/s}$$

In this example, Fiona has a speed v = 0.25 lt·s/s. This same speed could be expressed as $v = 0.25c$. In fact, we will typically express velocities as a fraction of the speed of light c.

D. Relativity of time intervals

The most startling consequence of the invariance of the speed of light is that it forces us to abandon the notion of absolute time. This means the time interval between two events depends on the velocity of the clocks used to measure the interval. The following *thought experiment* should help you understand this concept of the relativity of time intervals.

Imagine three identical clocks constructed as follows. Each clock contains a light source that emits a pulse of light toward a mirror some fixed distance away (see Figure 1.1(a)). The mirror reflects the pulse back toward the source. When the reflected pulse returns to the source and hits a triggering device, the source immediately fires a second pulse, which reflects from the mirror and triggers a third pulse, and so on. A count registers in a counter for each return pulse so the number of counts becomes a measure of elapsed time.

We place two of these light clocks, A and B, a fixed distance apart and at rest in a reference frame attached to the constant velocity Earth. We put the third clock, D, on a spaceship traveling at a constant speed v relative to the Earth (see Figure 1.1(b)), and perpendicular to the direction of travel of the light pulse in the clocks.

Suppose clock D emits a light pulse at the exact instant it passes clock A. Also suppose that the distance between A and B is such that clock D passes clock B at the precise instant clock D's reflected pulse returns to the source. We therefore have two events: Event #1 = "clock D passes clock A" and Event #2 = "clock D passes clock B." We label the time interval between these two events – measured by clock D – as Δt_{D}. The quantity Δt_{D} is called the *proper time* interval between the two events:

PROPER TIME IS DEFINED AS THE TIME MEASURED ON A SINGLE CLOCK THAT IS PRESENT AT BOTH EVENTS.

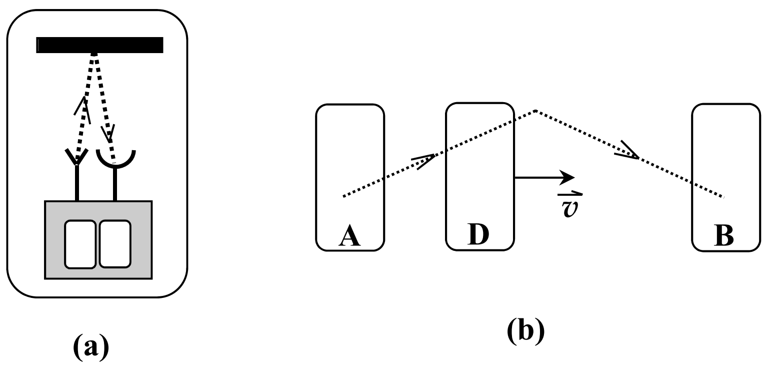

Figure 1.1 a) A light clock used in the thought experiment. b) Light clock D passes rest-frame clocks A and B. Dashed line shows path of D's light pulse as observed in A and B's rest frame.

In the case discussed above, clock D measures the proper time. In our particular arrangement, the proper time is exactly one tick.

We now pose the crucial question, the answer to which is the key to understanding all of special relativity:

What is the elapsed time Δt_{AB} as measured by clocks* A *and* B *for clock* D *to travel from* A *to* B*?

"Simple," you might think. "The answer is obviously exactly one tick, the same as that measured by clock D, right?" Not so simple. As we will see, the concepts of absolute time (i.e., everyone and everything measures the passage of time the say way) is a casualty of the invariance of the speed of light.

For the question posed above to have any meaning, clocks A and B must be synchronized; i.e., observers in the Earth's reference frame would say that the two clocks are reading the same time. (Note: this is not a trivial matter – we will discuss synchronization more fully in the next chapter.) The two-clock time Δt_{AB} is then the difference between the time reading on Clock A at Event #1 (clock D passes clock A) and the time reading on Clock B at Event #2 (clock D passes clock B).

In clock D's reference frame, clock A passes D first and then clock B passes D. The time interval between these events is Δt_{D} on clock D and therefore the light pulse in clock D travels a round-trip distance equal to $c \cdot \Delta t_{\text{D}}$. But from figure 1.1 this same pulse (the one inside clock D) travels a longer, zig-zag path when viewed from the frame in which clocks A and B are at rest. Because of the invariance of the speed of light, *this longer distance must translate into a longer time interval*. This means the round trip time for clock D's pulse is one tick as measured on clock D, but it is more than one tick when measured on clocks A and B. In other words, the elapsed time between the event "clock D passes clock A" and the event "clock D passes clock B" is *longer* when measured with the two clocks A and B than when measured with the single clock D.

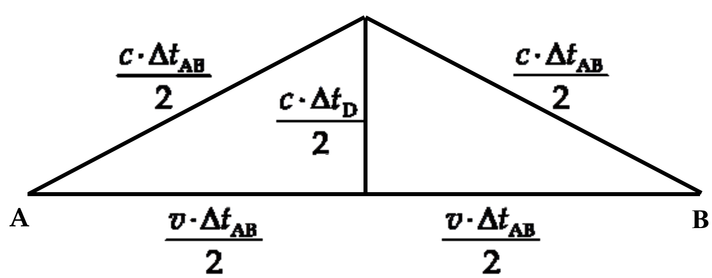

Figure 1.2 Derivation of the proper time relation. Δt_{AB} **= elapsed time on clocks A and B;** Δt_{D} **=elapsed time on clock D.**

How much longer is the two-clock time interval Δt_{AB} measured on the two clocks A and B than the proper time interval Δt_{D} measured on the single clock D which is present at both events?

We can find out by looking at the path taken by the pulse of light in clock D viewed from C's reference frame and from A and B's reference frame (see Figure 1.2). As we've already seen, in clock D's frame the pulse travels straight up and down along the vertical line in the figure and the total round-trip distance is $c \cdot \Delta t_{\text{D}}$. The same pulse, traveling for time Δt_{AB} relative to A and B travels the total zigzag distance $c \cdot \Delta t_{\text{AB}}$. Clock D itself travels a distance $v \cdot \Delta t_{\text{AB}}$ relative to clocks A and B while the pulse makes one round-trip in D.

Therefore, using the Pythagorean theorem on either small triangle in Figure 1.2, we find

$$\left(\frac{c \cdot \Delta t_{\text{AB}}}{2}\right)^2 = \left(\frac{c \cdot \Delta t_{\text{D}}}{2}\right)^2 + \left(\frac{v \cdot \Delta t_{\text{AB}}}{2}\right)^2$$

from which we solve for the proper time to obtain

$$\Delta t_{\mathrm{D}} = \Delta t_{\mathrm{AB}} \sqrt{1 - \frac{v^2}{c^2}} = \Delta t_{\mathrm{AB}} \sqrt{1 - \left(\frac{v}{c}\right)^2} \ .$$

This relation can be written in the general form:

$$\boxed{\Delta t_{proper} = \Delta t_{two-clock} \sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad (1.1)$$

This very important relation is sometimes called the "proper time relation" or the principle of "time dilation." Qualitatively, it expresses the fact that ***different observers measure the passage of time differently, depending on their relative motion and on the interval measured.*** And recall that our starting points were these three simple things: (1) the invariance of the speed of light; (2) that distance = (speed) × (time); and (3) the Pythagorean theorem.

Hidden inside Equation (1.1) is another result from special relativity; namely, no object can travel at a speed greater than c relative to any other object or reference frame. A superluminal speed ($|v| > c$) would result in an imaginary proper time, something that has no physical meaning. This speed limit is imposed by energy considerations as well (it would take an infinite amount of energy to accelerate an object with mass[1] to a speed $v = c$ relative to an observer, and *more* than an infinite amount of energy to achieve a speed $v > c$). Therefore, because $|v| \le c$, the proper time interval Δt_{proper} between two events is always *smaller* than the two-clock time interval $\Delta t_{two-clock}$ measured in a frame that requires two synchronized clocks for measurement.

Example 1.4. **Time dilation.** Anthony and Jenna are traveling on a train that moves with a constant speed of $1.8\text{x}10^8$ m/s (= $0.6c$) relative to the ground. They pass a parked VW Beetle at which point the two of them simultaneously yell, "Punch Buggy Red!" and poke each other on the shoulder. Three seconds later, Jenna yells, "Jinx!" What is the time between these two events according to Melody, who is inside the parked VW?

Solution: The key question – does anyone measure the proper time? To answer this, write this down in terms of events.
Event A = Anthony/Jenna poke each other; Event B = Jenna jinxes Anthony. In this example, Anthony and Jenna are at both events (not Melody in the VW), so Anthony and Jenna measure the proper time interval, which was given to be 3 s. So, we are

[1] A photon of light can be considered an "object" that travels at a speed c, but this is a massless object.

given Δt_{proper} and are asked to solve for $\Delta t_{two-clock}$, which is the time interval measured by the person in the car. We use equation (1.1):

$$\Delta t_{proper} = \Delta t_{two-clock}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\rightarrow \Delta t_{train} = \Delta t_{VW}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\rightarrow \Delta t_{VW} = \frac{\Delta t_{train}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{3\text{ s}}{\sqrt{1-\left(\frac{0.6c}{c}\right)^2}}$$

$$= \frac{3\text{ s}}{\sqrt{1-(0.6)^2}} = \frac{3\text{ s}}{\sqrt{1-0.36}} = \frac{3\text{ s}}{\sqrt{0.64}} = \frac{3\text{ s}}{0.8}$$
$$= 3.75\text{ s}$$

In this example, the people on the train measured the proper time because they were at both events, so the time interval is smaller from their reference frame. Be careful, though: sometimes the observer described as stationary measures the smaller time interval – it all depends on what the events are and who happens to be present at both of them. Note also that we expressed v as a fraction of the speed of light – it makes things a lot simpler to write v in this manner, as discussed previously.

Think About This #1.2:
In the previous example, the proper time is smaller than the two-clock time. Will the proper time always be smaller than the two-clock time, or will this depend on the situation?

Think About This #1.3:
In the previous example, it is clear that Anthony and Jenna on the train were at both events, so measured the proper time. We identify Melody to measure the two-clock time. But what does it mean to say that Melody measured the two-clock time?

E. Relativity of distance measurements

One thing that we have encountered repeatedly is the claim that relativity breaks down the distinction between distance and time. In fact, in relativity, distance and time are really just flip sides of the same coin. And as we will see now, you can't change our conception of time without making a similarly dramatic change in the way we view distances and length.

Consider the following thought experiment: a train is moving on a track, with observers Emmy and Maria at the front and back end of the train. Emmy and Maria have measured the length of the train with a long tape measure that they carry with them on the moving train, and find the length to be L_{train}. The train goes past observer Sophie who is standing next to the track with a stopwatch (see figure 1.3). Relative to Sophie in the "track reference frame", the train (and Emmy and Maria) is moving with a speed v. From the "train reference frame", of course, it is Sophie and the track that are moving at a speed v in the opposite direction.

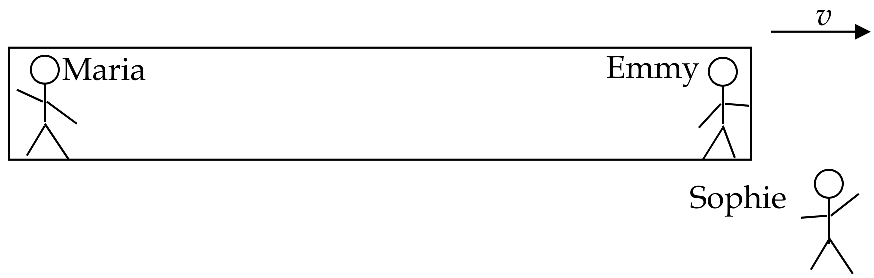

Fig. 1.3 Sketch of train for relativity of length thought-experiment.

Let's say that Sophie wants to measure the length of the train. She can use her stopwatch to do this: since distance = (speed) × (time), the length of the train is simply the speed v times the time interval between when the front of the train passes and when the back of the train passes. Let's consider two events: Event A = Emmy at the front of the train passes Sophie; Event B = Maria at the back of the train passes Sophie.

According to people on the train, the time between the two events $\Delta t_{\mathrm{AB}} = \dfrac{L_{train}}{v}$, where L_{train} is the previously-measured length of the train as measure by Emmy and Maria who are at rest with respect to the train – this is how far Sophie moves between the events according to train observers.

Think About This #1.4:
Why do the observers on the train measure the time between the two events to be the length of the train divided by the speed?

As we saw in the previous section, Sophie measures the proper time (as she is at both events), which is a smaller time interval than the two-clock time measured by the train observers:

$$\Delta t_{proper} = \Delta t_{two-clock}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

$$\Delta t_{Sophie} = \Delta t_{\mathrm{AB}}\sqrt{1-\left(\frac{v}{c}\right)^2}$$

Based on this result, Sophie determines the length of the train via distance = (speed) × (time) to be:

$$
\begin{aligned}
L_{according\ to\ Sophie} &= v \cdot \Delta t_{Sophie} \\
&= v \cdot \Delta t_{\mathrm{AB}} \sqrt{1-\left(\frac{v}{c}\right)^2}
\end{aligned}
$$

And since, from above, we know that $v \cdot \Delta t_{\mathrm{AB}} = L_{train}$, we obtain:

$$L_{according\ to\ Sophie} = L_{train} \sqrt{1-\left(\frac{v}{c}\right)^2}$$

The length of the train as measured by an observer by the side of the track is less than the length of the same train as measured by people moving with the train.

We can write this relation (referred to as the ***Lorentz contraction*** or simply ***length contraction*** equation) in more general terms:

$$\boxed{L_{other} = L_{rest} \sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{1.2}$$

where L_{rest} is the length of an object as measured by observers in a reference frame where that object is at rest, and L_{other} is the length as measured by observers in a different reference frame. Note that an object is always largest when measured from its own reference frame, and shorter when it is measured in a reference frame in which it is moving.

Example 1.5. **Length contraction.** Sam and Allen are standing on a train and determine the length of the train to be 200 m. Alex and Oliver, at rest in the track reference frame, observe that Sam, Allen and the train move past them at constant velocity v. Alex and Oliver determine the length of that same train to be 120 m. What is the velocity v of the train as observed by the track observers?

Solution: The key question – does anyone measure the rest length? Be careful here – though Alex and Oliver are described as at rest in the problem statement, they are not at rest with respect to the object being measured. As described above, the rest length of the train is the length of the train as measured by observers in a reference frame where the train is at rest; in this case, Sam and Allen measure the rest length of the train. We use equation (1.2) and solve for velocity:

$$L_{other} = L_{rest}\sqrt{1-\left(\frac{v}{c}\right)^2} \Rightarrow \frac{L_{other}}{L_{rest}} = \sqrt{1-\left(\frac{v}{c}\right)^2} \Rightarrow \left(\frac{L_{other}}{L_{rest}}\right)^2 = 1-\left(\frac{v}{c}\right)^2$$

$$\Rightarrow \left(\frac{v}{c}\right)^2 = 1-\left(\frac{L_{other}}{L_{rest}}\right)^2 \Rightarrow \frac{v}{c} = \sqrt{1-\left(\frac{L_{other}}{L_{rest}}\right)^2} \Rightarrow v = \left(\sqrt{1-\left(\frac{L_{other}}{L_{rest}}\right)^2}\right)\cdot c$$

$$\Rightarrow v = \left(\sqrt{1-\left(\frac{L_{according\ to\ Alex/Oliver}}{L_{according\ to\ Sam/Allen}}\right)^2}\right)\cdot c = \left(\sqrt{1-\left(\frac{120}{200}\right)^2}\right)\cdot c = \left(\sqrt{1-(0.6)^2}\right)\cdot c$$

$$v = \left(\sqrt{1-0.36}\right)\cdot c = \left(\sqrt{0.64}\right)\cdot c = 0.8c$$

Think About This #1.5:
If we had incorrectly identified the L_{rest} and the L_{other} measurements, what would we calculate for the velocity? What general conclusions can we make about the size of L_{rest} compared with the size of L_{other} ?

Several comments are in order. (1) You can't have time dilation without length contraction → the two necessarily go hand-in-hand. This is a recurring theme of relativity – Einstein's theory can't be taken "a la carte"; rather, it is all or nothing. Einstein realized that if any single prediction of relativity were ever refuted, then the entire theory would have to be discarded. (2) Length contraction is not an illusion or merely a matter of perception. In the example, the length of the train doesn't just *appear* to be smaller in the track's reference frame; rather, it ***really is smaller*** in that reference frame.

F. Relativistic Velocity Transformations

Let's say that two spaceships leave Earth. The *USS Zaphod* leaves the Earth going in one direction with a speed 0.8*c* relative to the Earth. The *USS Beeblebrox* leaves the Earth going in the opposite direction with a speed –0.8*c* relative to the Earth. How fast is the speed of the *Zaphod* from the reference frame of the *Beeblebrox*? Based on classical principle of velocity addition, you might expect the answer to be 1.6*c*. But this conflicts with a result that we deduced from Einstein's theory of relativity which states that no object can travel faster than the speed of light relative to *any* other observer (see the discussion preceding Example 1.4). It is clear that it is necessary to replace classical laws for addition and subtraction of velocities with a more general, relativistic transformation.

The need for a relativistic approach to velocity addition and subtraction was already hinted at earlier. The principle of invariance of the speed of light requires that all observers (in any reference frame) measure the same speed for a pulse of light. So, we can't simply add and subtract velocities. On the other hand, common experience shows us that for non-relativistic speeds, simple addition and subtraction works fine. So, we need a velocity transformation relation that reduces to the classical result for small speeds, but which prevents anything from traveling faster than the speed of light. It turns out that this can be accomplished by taking the classical result and dividing by a relativistic correction that becomes significant (i.e., not just 1) for speeds close to c.

Figure 1.4 shows the scenario that we are discussing. Two reference frames are defined: an unprimed frame denoted by observer A and a primed frame denoted by observer B on a spaceship moving with a speed v relative to observer A. They are both measuring the speed of the same object. Observer A says the object is moving with a speed u_{obj} while observer B says the ball is moving with a speed u'_{obj}. By convention, we'll use v for the speed of reference frames and u for the speed of objects as measured within reference frames.

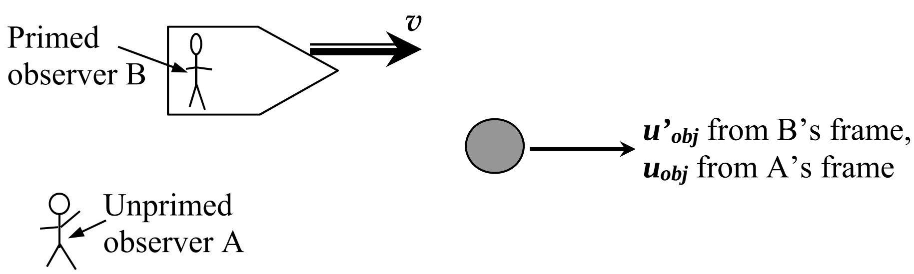

Figure 1.4 An object moving in the x-direction relative to both unprimed and primed frames. The speed of the object is measured to be u_{obj} from the reference frame of observer A and u'_{obj} from the reference frame of observer B.

The question is how u_{obj}, u'_{obj} and v are related. The result is a *velocity transformation* equation,

$$\boxed{u_{obj} = \frac{u'_{obj} + v}{1 + u'_{obj} v / c^2}} \tag{1.3}$$

Equation (1.3) can be used to relate an object's velocity in one frame to that as viewed in another frame.

We won't derive equation (1.3) rigorously here. Rather, note that if the object is a pulse of light, then $u'_{obj} = c$, and equation (1.3) reduces to:

$$u_{obj} = \frac{c + v}{1 + cv / c^2} = \frac{c + v}{1 + v / c} = c\frac{c + v}{c + v} = c\,.$$

Both observers measure the object to have a speed c, so, the invariance of the speed of light is preserved in this transformation.

Note that the numerator in (1.3) is the result that you would get classically, whereas the denominator is a relativistic correction. Also, note that if either the object or the primed observer are traveling at speeds that aren't a reasonable fraction of the speed of light, then the denominator of (1.3) is very nearly 1, so we recover the classical result for "everyday" speeds.

We'll show how to work with this relation in the next example.

Example 1.6. Nacia throws a *really* fast fastball while riding on a *really* fast train. From her reference frame (i.e., the train's frame) the ball moves toward the front of the train with a speed $u'_{ball} = 0.7c$. The train itself is moving relative to the ground with speed $v = 0.8c$. How fast is the ball moving relative to someone on the ground?

Solution. Classically, the speed as viewed from the ground would be $u'_{ball} + v$ or $1.5c$. (This is the numerator of Eq. (1.3).) But, of course, this isn't possible in a universe where nothing goes faster than c. Using Eq. (1.3):

$$u_{ball} = \frac{u'_{ball} + v}{1 + u'_{ball} v / c^2} = \frac{0.7c + 0.8c}{1 + (0.7)(0.8)} = 0.96c$$

Note that the relativistic correction keeps the speed less than c.

Think About This #1.6:
What if instead of a ball, Nacia "threw" a beam of light? How fast would it be moving from the train's reference frame? How fast from the ground's reference frame?

If a problem gives you the speed as measured by the unprimed observer, you can use the following inverse transformations to get the speed as measured by the primed observer:

$$\boxed{u'_{obj} = \frac{u_{obj} - v}{1 - u_{obj} v / c^2}} \qquad (1.4)$$

Please don't panic trying to keep track of u's and v's and primes and unprimes. You don't really need to write down these relations or try to figure out which speed is v, which speed is u_{obj} and which speed is u'_{obj}. There is a very simple way of handling all of these problems. No matter which velocity you are looking for, the answer is always:

$$\frac{\text{classical result}}{\text{relativistic correction}},$$

where the relativistic correction is simply "1 + (product of other two speeds, without the c's)" or "1 – (product of other two speeds, without the c's)". You will be given two velocities, and you'll be looking for the third one. Just figure out the answer classically, then divide by the correction. The only question then is whether to use the "+" or "–" in the correction. The rule: if you *added* magnitudes of velocities in the numerator, then you use the "+" in the denominator, and if you *subtracted* magnitudes in the numerator, then you use the "–" in the denominator. This will take care of any velocity addition or subtraction that you need.

G. Experimental evidence

Most people are understandably skeptical when they first read about the predictions of special relativity. This is to be expected, since we do not experience time dilation or length contraction effects on a daily basis. For these effects to be significant, you need relative velocities that are significant fractions of the speed of light. Looking at both equations (1.1) and (1.2), the key piece is the factor $\sqrt{1-\left(\frac{v}{c}\right)^2}$, which is almost identically equal to 1.00 for even the fastest velocities that people ever experience. This is an important of an aspect of relativity; namely, that it obeys ***classical correspondence***, i.e., the results of relativity agree with Newton's classical results for smaller velocities.

Despite the fact that relativistic effects are almost negligible in personal "everyday" phenomena, there is ample experimental evidence that shows that Einstein's predictions are correct. In *every* case where an experiment has tested the theory of relativity, the experimental results have *always* agreed precisely with the predictions of relativity. Some examples:

- **Time dilation**. Time dilation is the most tested aspect of relativity. The most direct test was performed by taking two identical atomic clocks, flying one around the world on a plane and leaving the other on the ground, then comparing their readings after the trip. As predicted by Einstein, the clocks had ticked off different times, and by precisely the predicted amount.[2] Particle decay has also been used to test time dilation: a type of particle that typically lives for a certain period of time has been shown to live significantly longer if accelerated to high speeds (relative to the ground); again, the difference in times agrees perfectly with relativity. And the Global Positioning System (GPS) – which involves a series of satellites with precise clocks – uses relativity extensively to keep the orbiting clocks synchronized with those in the GPS units on the Earth. Without relativistic corrections, GPS wouldn't work!

- **The speed of light as a speed limit.** This result is verified daily in particle accelerators. It is fairly straightforward for scientists to accelerate subatomic particles to speeds close to the speed of light. But no matter how much energy is added, the speeds never make it to or above c. Electrons, in particular, have been accelerated to speeds > $0.9999c$, but never up to or above c.

- **Length contraction**. No experimentalist has managed to accelerate a train to relative speeds large enough to measure length contraction effects. (Trust us: you wouldn't want to be anywhere near a train going this fast.) But there *is* experimental evidence for length contraction: (a) cosmic rays produced at the top of the Earth's atmosphere somehow manage to make it to the surface of the Earth before decaying, despite the fact that they are *very* unstable. Some of these particles have lifetimes so short that even traveling at speeds close to c, they would be expected to decay long before they reach the ground. This can be explained using length contraction: the distance from the top of the atmosphere to the Earth's surface is significantly contracted from their reference frame, so there is no problem making it to the Earth's surface before decaying.[3] (b) Another piece of experimental evidence comes from electromagnetic theory – it turns out that you can explain why an electrical current produces magnetic effects by applying relativistic length contraction to the stream of electrons. The argument is too long to present here (especially since we haven't covered electricity and magnetism yet), but suffice it to say that the results agree perfectly with an analysis based on length contraction.

There are other experimental tests of other aspects of relativity. But, in general, it is worth remembering that relativity is not a series of different theories, but rather is a single, coherent, internally consistent theory. All of the predictions are inherently related to each other. So you can't say, "Well, I'm fine with time dilation but I don't buy length contraction." You simply can't have time dilation without length contraction – they are the same thing.

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[2] Note that General Relativity plays a role here since the reference frames aren't rigorously inertial, but the experiments took account of these general relativistic effects.

[3] This result can also be explained using time dilation, of course, since time dilation and length contraction are really the same thing.

PROBLEMS

1) This problem gives you practice with converting units associated with speed.
   a) A proton is traveling at a speed of 4.0×10^7 m/s. How many lt•s/s is this?
   b) A sub-atomic particle is traveling at 0.060 lt•min/min. Convert this to m/s.

2) You've likely noticed the importance of the factor $\sqrt{1-(v/c)^2}$.
   a) Show that for $v = 0.6c$, then $\sqrt{1-(v/c)^2} = 0.8$.
   b) When $v = 0.8c$, what is $\sqrt{1-(v/c)^2}$?
   c) When $\sqrt{1-(v/c)^2} = 0.5$, what is v ?

3) Duck Dodgers hops in his spaceship and leaves the Earth at a constant velocity of 0.6*c* in an attempt to reach the newly discovered Planet X before aliens from Mars.
   a) Mission control on Earth sends an encoded message (a flashing beacon) to Duck Dodgers warning him about the progress of the Martian ship. The light pulses travel at a speed *c* relative to observers on the Earth. How fast are the pulses traveling relative to Duck Dodgers?
   b) Duck Dodgers doesn't understand the message that he received, so he sends a radio wave message back toward the Earth asking for clarification. The radio signal is traveling at a speed *c* relative to the Duck. How fast is the signal traveling relative to observers on the Earth?
   c) The radio message is intercepted by Marvin the Martian who is behind Duck Dodgers but traveling in the same direction at a speed 0.8*c* relative to the Earth. How fast is the radio message relative to Marvin?
   d) At this point, you might be asking yourself "What was the point of this problem?" What *was* the point of this problem?

4) Charlie travels at speed 0.50*c* to the right relative to Kyle. Zel is traveling at 0.70*c* to the left relative to Kyle. Calculate Charlie's velocity relative to Zel.

5) Leah and Daniel are returning home, traveling at a speed 0.75*c* relative to and toward the Earth. Leah is particularly anxious to get back to Earth, so Leah hops on the emergency shuttlecraft, which leaves the ship traveling at a speed of 0.75*c*, relative to Daniel. How fast is Leah traveling relative to the Earth?

6) The Road-Runner travels to the right at 0.8*c*, and is chased by the Coyote moving at 0.6*c*. How fast and in what direction is the Road-Runner traveling, according to the Coyote? How fast and in what direction is the Coyote traveling, according to the Road-Runner?

7) Jamie's speed is measured at +0.6*c* relative to Joseph, who is at rest on the earth, and +0.8*c* relative to Mia, who is passing by in a rocket. Determine the speed of Mia's rocket relative to Joseph on the earth.

8) Ashe is sitting in a spaceship moving at constant velocity 0.80 lt·s/s. They pass between two planets A and B in 1000 s, as measured by synchronized clocks on the planets. Calculate the elapsed time according to a clock carried on Ashe's spaceship.

9) Jorisha is sitting on a train. How fast does Jorisha have to travel relative to Pyxie and Leah, who are at rest relative to each other, in order that the elapsed time as measured on Jorisha's clock is one-tenth the elapsed time measured by Pyxie and Leah?

10) Mina is on a meteorite that is observed to travel a distance 1.00×10^5 lt·s in a time of 6.00×10^5 s (distance and time measured relative to the Earth rest frame, velocity of the meteorite assumed to be constant). Calculate the elapsed time for this trip as measured by Mina.

11) Margaret is in a car stopped at a traffic light. She beeps the horn, and 2.4 μs later beep the horn again (1 μs = 1 microsecond = 10^{-6} s). What is the time between the two beeps as measured by Krishna, who is passing by the car in a ship moving at a constant velocity $0.8c$?

12) A muon is an unstable elementary particle which decays soon after being created. One such muon travels at $0.98c$ towards the earth. According to clocks in the muon's rest-frame, the muon lives 1.8×10^{-6} s (in other words, it decays 1.8×10^{-6} s after being created). How long does the muon live according to clocks in the Earth's rest-frame?

13) Harry travel in a rocketship moving at a constant velocity of $0.80c$ from Earth to a nearby star, Alpha Centauri, a distance of 4.00 lt·yr. Note: velocity and distance are measured according to the Earth/Alpha Centauri rest frame.
    a) How long does the trip take according to Ron, who is at rest relative to Earth?
    b) How long does the trip take according to Hermione, who is on Alpha Centauri and at rest relative to Alpha Centauri?
    c) Calculate the time for the trip as measured by Harry.
    d) Based on your answer to c), calculate the distance between Ron (on Earth) and Hermione (on Alpha Centauri), as determined by Harry using the relation distance = (speed) × (time), where distance, speed and time are all measured from their reference frame.
    e) Calculate the distance between Ron and Hermione from Harry's reference frame, but this time use length contraction. You should end up with the same result as for d). Hopefully, this will convince you that length contraction and time dilation are really the same thing (i.e., you can't have one without the other).

14) Sheldon (who loves trains) is standing next to a train track. A *really* long train moves by traveling *really* fast ($0.9c$, in Sheldon's frame). When the front of the train passes by him, Sheldon starts his stopwatch. When the back of the train passes by him, Sheldon stops his stopwatch and notices that 0.0025 s have elapsed.
    a) According to Sheldon, how long is the train?
    b) How long is the train according to Amy and Penny (Penny, Penny), who are sitting on it?

15) There is a supergiant star named Betelgeuse which (from the Earth's reference frame) is 80 lt·yr away (assume the distance between Earth and Betelgeuse does not change for the situations described below). Betelgeuse is located in the constellation Orion and could go supernova[4] anytime in the next million years.
   a) John travels toward Betelgeuse at a constant speed $0.8c$ relative to the Earth-Betelgeuse reference frame. What is the separation between Earth and Betelgeuse in John's reference frame?
   b) Robin is traveling toward Betelgeuse, and measures the Earth-Betelgeuse distance to be 23 lt·yr. How fast is Robin traveling relative to the Earth?

16) A spaceship crew wants to make the trip from Earth to Alpha Centauri (4.00 lt·yr apart in the Earth/Alpha Centauri rest frame) in only 2.0 years as measured by clocks on board their spaceship which travels at constant velocity. Determine how fast the crew must travel relative to Earth. **note that this is a challenging problem, both conceptually and algebraically, given only the tools introduced in this chapter**

[4] That will be a very exciting event, though it will take 80 years for us to know, since that is the time it will take light from the event to reach the earth. We'll learn something about supernova in Chapter 4.

Relativity 2: Spacetime

Learning Goals

1. **Calculate a spacetime interval between two events, and classify the interval as space-like, time-like or light-like. Use these classifications to determine whether or not the two events are causally linked or can have their time-order or space-order reversed.**
2. **Use the invariance of the interval to relate distance and time intervals between two events in one reference frame to those in a different frame.**
3. **Draw a spacetime diagram based on descriptions and numerical values.**
4. **Determine the velocity of an object from the reciprocal of the slope of its worldline on a spacetime diagram.**
5. **Use a space-time diagram to determine whether the space-time interval between two events is space-like, time-like, or light-like.**
6. **Use a space-time diagram to determine time- and spatial-ordering of events, including whether or not events are simultaneous or at the same location in particular frames.**

Previously, we introduced the basic ideas of relativity along with some of the most dramatic implications of the theory. But the predictions of time dilation and length contraction are merely special cases of a much broader theory. In this chapter, we discuss the idea of *spacetime*, which blends time and space together. We introduce the *spacetime interval*, a quantity that is one of the fundamental invariants in relativity, and we use this interval to relate distance and time measurements between different observers. We also introduce *spacetime diagrams*, which provide a graphical way of illustrating relativistic phenomena, particularly the relativity of *simultaneity*.

A. Spacetime intervals

As we have seen, observers in different reference frames disagree about time and distance measurements. But there are a few quantities referred to as *invariants* upon which all observers agree regardless of their reference frames. One of these invariants was discussed in the previous chapter; namely, the invariance of the speed of light in a vacuum. It turns out that distance and time can be folded together to make another invariant, referred to as the invariant *spacetime interval* $(\Delta S)^2$, defined by

$$(\Delta S)^2 = (c\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2].$$

The term in square brackets might be familiar to you: it is the square of the distance between two points in a three-dimensional Euclidean space as determined by using the Pythagorean theorem. While you may not recognize it in three dimensions, you do know its two dimensional equivalent. So in some sense the equation above is like the

Pythagorean theorem in *four* dimensions. As we have been considering motion along a straight line, the equation above simplifies to

$$\boxed{(\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2} \tag{2.1}$$

where Δt is the time between two events, $c\Delta t$ is a distance (specifically the distance that light could travel in the time between those two events), and Δx is the distance between those same two events, all as measured in the same reference frame.

Think About This #2.1:
Why does limiting our consideration of motion to be along a straight line allow for $(\Delta S)^2 = (c\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]$ to simplify to $(\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2$?

Note that $(\Delta S)^2$ can be positive, zero or negative. If $(\Delta S)^2$ is positive, then the interval is called *time-like* since the first term – with Δt in it – dominates. Similarly, negative values of $(\Delta S)^2$ correspond to *space-like* intervals, and if $(\Delta S)^2 = 0$, the interval is called *light-like*. Qualitatively, an event is light-like if a pulse of light could travel directly between the two events. This can be seen from Eq. (2.1): if $(\Delta S)^2 = 0$, then

$$\begin{aligned} &0 = (c\Delta t)^2 - (\Delta x)^2 \\ &\rightarrow \Delta x = \pm c\Delta t \end{aligned}$$

as would be expected for a pulse of light traveling a distance Δx in a time Δt.

Two events could be ***causally-linked*** (i.e., event A actually causes or contributes to event B) if the spacetime interval between them is either light-like or time-like. In fact, for a time-like event, the spacetime interval is the proper time. If two events are separated by a space-like interval, then no information can travel between the two events since it would require superluminal ($v > c$) information transmission, and nothing (especially information) can travel faster than light relative to any observer. So events *can't* be causally linked if the square of the spacetime interval between them is negative.

Example 1. Causality and intervals. In the year 2030, a mother and her daughter are watching the 7th game of the World Series from a Moon base at the Sea of Tranquility. The daughter sneezes and then watches in horror as 2.0 s later Derek Jeter, Jr., of the Boston Red Sox strikes out with the bases loaded to end the series. Distraught, the daughter bursts into tears. "What's wrong?" her mother asks. "It's my fault that the Red Sox lost! My sneeze caused Jeter to strike out!" What argument should the mother use to assure her daughter that she is not personally responsible for *yet another* heart-breaking[1] Red Sox loss?

[1] Sports fans might like to know that this example was written in 2003, before the curse was broken.

Solution: The mother should first point out that the distance between the Earth and Moon is $3.84\text{x}10^8$ m = 1.3 lt·s. As the mother & daughter received the TV signal of the strikeout 2.0 s after the sneeze, in their reference frame the strikeout must have actually occurred only 0.7 s after the sneeze. (It takes the TV signal 1.3 s to make it from the Earth to the Moon.)

Now, the mother should calculate the spacetime interval:

$$\begin{aligned}(\Delta S)^2 &= (c\Delta t)^2 - (\Delta x)^2 \\ &= ((1 \text{ lt}\cdot\text{s/s})(0.7 \text{ s}))^2 - (1.3 \text{ lt}\cdot\text{s})^2 \\ &= 0.49 \ (\text{lt}\cdot\text{s})^2 - 1.69 \ (\text{lt}\cdot\text{s})^2 \\ &= -1.2 \ (\text{lt}\cdot\text{s})^2\end{aligned}$$

So, the mother should pat the daughter on the head and say, "So, you see honey – you ***can't*** have caused Jeter to strike out because the spacetime interval between your sneeze and his strikeout is a space-like interval!" (That should be very comforting to all concerned).

Don't worry about the fact that $(\Delta S)^2$ is negative for space-like intervals. The definition of $(\Delta S)^2$ in equation (2.1) is chosen so that for time-like intervals, the interval ΔS itself is the proper time (as we'll discuss later), which is convenient for our purposes. But the interval could have equally been defined as $(\Delta S)^2 = (\Delta x)^2 - (c\Delta t)^2$ in which case $(\Delta S)^2$ would be negative for *time-like* intervals (some people do, in fact, define $(\Delta S)^2$ this way). In fact, one could define two different intervals: $(\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2$ for time-like intervals and $(\Delta S)^2 = (\Delta x)^2 - (c\Delta t)^2$ for space-like intervals. We will stick with equation (2.1).

As stated earlier, $(\Delta S)^2$ is an invariant – observers in different reference frames will agree on the value of this interval for any two events:

$$\boxed{\begin{array}{ccc}(\Delta S)^2 & = & (\Delta S')^2 \\ (c\Delta t)^2 - (\Delta x)^2 & = & (c\Delta t')^2 - (\Delta x')^2\end{array}} \qquad (2.2)$$

where Δx and Δt are the distance and time between two events as measured in one reference frame, and $\Delta x'$ and $\Delta t'$ are the distance and time between the same two events as measured in another reference frame.

Think About This #2.2:
What does it mean for a quantity to be *invariant*? What are two invariant quantities we have covered so far in our relativity reading?

The invariance of the spacetime interval is important for several reasons. First, the invariance of the interval helps to further clarify the intimate connection between distance and time in relativity. Any disagreements between different observers about the time interval between events must be accompanied by a corresponding disagreement in the distance in order to keep the interval invariant. Second, if an interval is space-like or time-like or light-like as viewed in one reference frame, then it is the same kind of interval as viewed in any reference frame. This makes sense: if two events cannot be causally-linked in one reference frame, for instance, it would be nonsensical to think that they would be causally-linked as observed by someone in a different reference frame. Finally, the interval can be used to determine how events are viewed in one reference frame, given information in a different frame. As an example, we refer back to a problem from a previous assignment, in Example 2 below.

Think About This #2.3:
Pause for a moment to consider the bolded passage above. In your own words, explain why the invariance of the spacetime interval is useful/important.

Example 2. Using the interval. A spaceship crew wants to make the trip from Earth to Alpha Centauri in only 2.0 years as measured by clocks on board their spaceship. (a) How long does the trip take according to Earth-frame observers? (b) How fast must the astronauts travel relative to Earth?

Solution: You might have done this problem in the previous chapter, where it was conceptually and algebraically challenging. Here, we do the same problem but using the spacetime interval, which makes it less complicated.

(a) Let's begin by identifying events. We have the Event S/E where the Spaceship and the Earth pass each other. We also have the Event S/AC, where the Spaceship and Alpha Centauri pass each other. We'll call the Earth/Alpha Centauri reference frame the unprimed frame, and the Spaceship the primed frame.

We'll begin by looking at the events from the reference frame of the Earth. We know from previous problems that the distance to Alpha Centauri is 4 lt·yr as measured by observers on the Earth, so the distance between the events S/E and S/AC is given by Δx = 4 lt·yr, according to observers in the Earth reference frame. We don't know the time between the events S/E and S/AC in this frame (that's one of the things we're looking for, so Δt is unknown.

Next, we'll look at the events from the reference frame of the Spaceship. From the statement of the problem, we can see that we're given the time between the two events in this frame, so $\Delta t' = 2$ yr. What about $\Delta x'$? **Since the Spaceship is present at both events, it follows that $\Delta x' = 0$.**

Using the interval and its invariance equation (2.2), we have

$$(\Delta S)^2 = (\Delta S')^2$$
$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2$$

$$(c\Delta t)^2 = (c\Delta t')^2 - (\Delta x')^2 + (\Delta x)^2$$
$$(c\Delta t)^2 = ((1\ \mathrm{lt}\cdot\mathrm{yr/yr})(2\ \mathrm{yr}))^2 - (0)^2 + (4\ \mathrm{lt}\cdot\mathrm{yr})^2 = 20\ (\mathrm{lt}\cdot\mathrm{yr})^2$$

$$(c\Delta t) = \sqrt{20\ (\mathrm{lt}\cdot\mathrm{yr})^2} = 4.4721\ \mathrm{lt}\cdot\mathrm{yr}$$

$$\Delta t = \frac{4.4721\ \mathrm{lt}\cdot\mathrm{yr}}{c} = \frac{4.4721\ \mathrm{lt}\cdot\mathrm{yr}}{1\ \mathrm{lt}\cdot\mathrm{yr/yr}} = 4.4721\ \mathrm{yr}$$

(b) Since we know the distance the Spaceship traveled and the time it took to make the trip, all in the same reference frame, we can determine its speed. The speed of the astronauts' ship is the distance it traveled divided by the time it traveled, where we have now ensured that the distance and time are measured in the same frame.

$$v = \frac{\Delta x}{\Delta t} = \frac{4\ \mathrm{lt}\cdot\mathrm{yr}}{4.4721\ \mathrm{yr}} = 0.894\ \mathrm{lt}\cdot\mathrm{yr/yr} = 0.894c$$

As you might have guessed from this example, the relations from the previous chapter (time dilation and length contraction) are both special cases of the more general invariance of the spacetime interval. The proper time relation equation from the previous chapter corresponds to a situation where one of the observers is at both events.

Let's say that the observer in the primed reference frame is at both events (i.e., measures the proper time), so $\Delta t' = \Delta t_{proper}$. Since this observer is at both events, clearly she measures the distance between the events to be zero, so $\Delta x' = 0$. In the unprimed frame, the distance between events Δx is simply $\Delta x = v\Delta t$ (distance = speed × time). Using the interval and its invariance, equation (2.2), we obtain:

$$\begin{aligned}
(\Delta S)^2 &= (\Delta S')^2 \\
(c\Delta t)^2 - (\Delta x)^2 &= (c\Delta t')^2 - (\Delta x')^2 \\
(c\Delta t)^2 - (v\Delta t)^2 &= (c\Delta t')^2 - (0)^2 \\
(c\Delta t')^2 &= (c\Delta t)^2 - (v\Delta t)^2 \\
(c\Delta t')^2 &= (\Delta t)^2[c^2 - v^2] \\
(\Delta t')^2 &= (\Delta t)^2 \frac{[c^2 - v^2]}{c^2} = (\Delta t)^2\left[1 - \left(\frac{v}{c}\right)^2\right] \\
\Delta t' &= \Delta t\sqrt{1 - \left(\frac{v}{c}\right)^2}
\end{aligned}$$

which is, in fact, the proper time relation (1.1) with $\Delta t'$ as the proper time (since the primed observer is at both events) and with Δt as the two-clock time. Similar arguments can be used to show that the length contraction relation equation is a special case of the invariance of the spacetime interval for situations where Δt or $\Delta t'$ is zero.

B. World Lines and Spacetime Diagrams

The motions of particles, clocks, or whatever can be represented on a spacetime diagram. A spacetime diagram consists of a pair of perpendicular axes, with the vertical axis representing time and the horizontal axis representing x-position in an unprimed inertial reference frame. The x-axis is the direction of relative motion between this unprimed frame and another inertial frame called the primed frame.

A plot of an object's position vs. time on a spacetime diagram is called the *world line* of the object. Three world lines are shown in figure 2.1; a straight world line represents motion with constant velocity while a curved world line represents accelerated motion. An *event* is represented by a dot on the spacetime diagram.

When drawing a spacetime diagram, make sure you use appropriate units. (Do *not* use meters for length!) For time (which we use as the vertical axis on a spacetime diagram), we choose something appropriate to the time scale of the problem, like years, seconds, nanoseconds, etc. Then we must choose an appropriate unit of distance equal to that traveled by light in the chosen unit of time. For example, suppose we choose one second as the unit of time, then we would use one lt·s (the distance traveled by light in one second) as the unit of distance. In these units the speed of light is c = (1 lt·s) / (1 s) = 1 lt·s/s. In this method of handling the units, the world line for a pulse of light must have a slope that is numerically equal to +1 or –1. ***And no world line can ever have a slope with magnitude less than 1 → that would correspond to an object traveling faster than light.*** Slopes can also be used to determine if the interval between two events is time-like, light-like or space-like. If a line were to be drawn connecting the two events, a time-like interval would correspond to a slope with magnitude greater than 1, a light-like interval would correspond to a slope with magnitude 1, and space-like interval would correspond to a slope with magnitude less than 1 (note that in this last case, that "line" drawn between the two events can't be the world line of any real object, since nothing can travel faster than c).

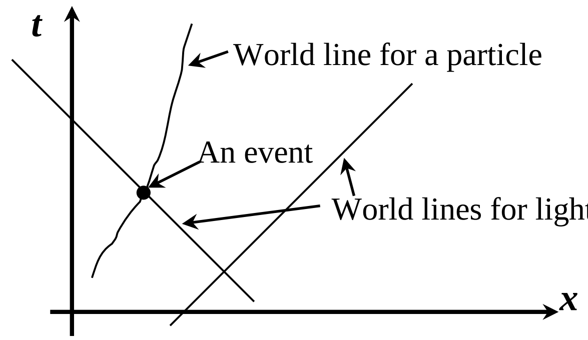


Figure 2.1 A spacetime diagram with three world lines. The two world lines for light have slopes +1 and –1.

Let's use a spacetime diagram to display the world lines of the three-clock thought experiment of Ch. 1, Section C (see Figure 1.1 from that section and Figure 2.2 below). For example, put clock A at rest at $x = 0$ and clock B at rest at $x = 0.60$ lt·s. The world lines for the stationary clocks A and B are then vertical lines at $x = 0$ lt·s and $x = 0.60$ lt·s. Let clock C travel with speed $0.60c$ in the positive x-direction. Because $c = 1$ lt·s/s, clock C passes through $x = 0$ at time $t = 0$ s, and it passes through $x = 0.60$ lt·s at time t = 1.0 s. It has traveled a distance of 0.60 lt·s in a time 1.0 s.

Notice in figure 2.2 that we have labeled the world line of clock C as the t' axis. This is a general result: the world line of a particular observer (say, someone traveling in a space ship) is the t' axis for that observer. This can be understood by considering a person on a spaceship holding a ball. The world line for the ball is the same as the world line of the ship and person since they are all moving together. From the perspective of the astronaut, the ball remains right in front of him and isn't moving anywhere, so it makes sense that that astronaut will say that the location of the ball remains at $x' = 0$. And just as it is true that the points where $x = 0$ in the unprimed frame define the t-axis, so it is that the points where $x' = 0$ in the primed frame define the t'-axis.

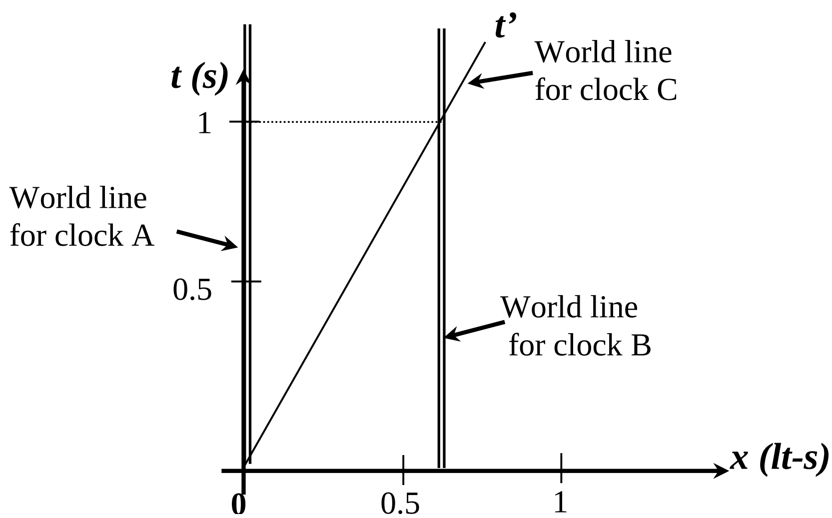


Figure 2.2 World lines for the three clocks in the thought experiment of section A.

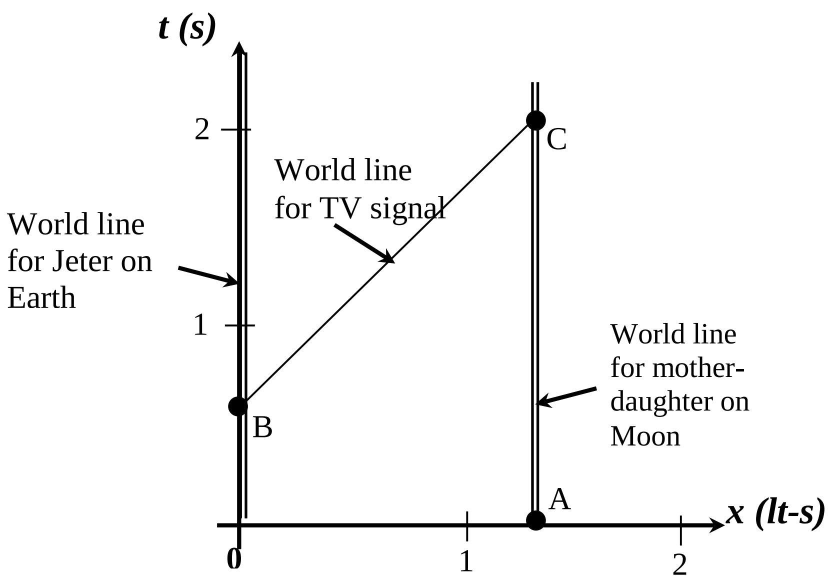
Some comments are in order:

1. A world line is nothing more than a plot of position versus time. If you ever find yourself stumped about how to plot a world-line, ask yourself: "Where is the (whatever) at time $t = 0$ (i.e., what is its x value)? Where is it at time $t = 1$? At time $t = 2$? …" Then simply plot those points and connect them with a line.
2. The slope of a world line is simply $1/v$. Practically, this means that if you have a ship moving at a speed of, say, $0.5c$, then the slope will be $1/v$ or 2.0 s/lt·s. When plotting a world line, this means that you go up 2 and over 1 (or over 0.5 and up 1).
3. Don't ***ever*** forget – nothing can travel faster than light, so there should ***never*** be a world line on your plot with a slope whose magnitude is less than 1.
4. Remember: events are plotted as dots.
5. Label everything clearly.

Example 3. Spacetime diagram corresponding to Example 1. Draw the spacetime diagram for the baseball scenario (Red Sox losing the World Series) shown in Example 1. Show the world lines for Derek Jeter, Jr., the girl and her mother and the TV signal. Also, show and label the following events: A – girl sneezes, B – Jeter strikes out, and C – girl and mother *see* Jeter striking out.

Solution : The world lines for Jeter and the girl/mother are simply straight vertical lines since they aren't moving in the Earth-Moon reference frame. If this isn't clear, then answer these questions: if we put the Earth at x = 0 at time t = 0, where is the Earth at time t = 1 s? Answer: still at x = 0. At t = 2 s? Answer: still at x = 0. The Earth's world line is nothing more that a set of points where x is always zero. As for the girl/mother on the Moon, we already said in Example 1 that they are about 1.3 lt·s away from the Earth.

We know from the problem that the girl/mother see the strikeout 2 s after she sneezes. So, if she sneezes at t = 0 (it is arbitrary as to what we choose as the t = 0 time), then the TV signal arrives at t = 2 s. It must have been sent from the Earth at an earlier time, and since it travels at the speed of light, then the world line for the TV signal is a 45° line. The only thing left is to plot the three dots for the events.



Note that if you imagine a line between A and B, that line would have a slope with magnitude less than 1 (i.e., too shallow), indicating that nothing can travel between these two events, consistent with the result in Example 1 that the interval is space-like and the corresponding events can't be causally-linked.

C. Ordering of events – the relativity of simultaneity

Every event has a set of space and time coordinates. In Example 3 above, we would say that the event A (girl sneezes) occurs at time t=0 and location x = 1.3 ls. Similarly, we can determine the location and times of events B and C, all as measured by observers in the Earth-Moon reference frame. Let's add one more event to the scenario: let's say that at time t = 0, the pitcher Pedro Martinez Jr., pitches the ball toward Jeter. In the diagram below, we have added this event and labeled it P. In the Earth-Moon frame, we can say quite definitively that A and P are simultaneous and come first, then B, then C. Also, A and C happen at the same location, and P and B happen at the same location.

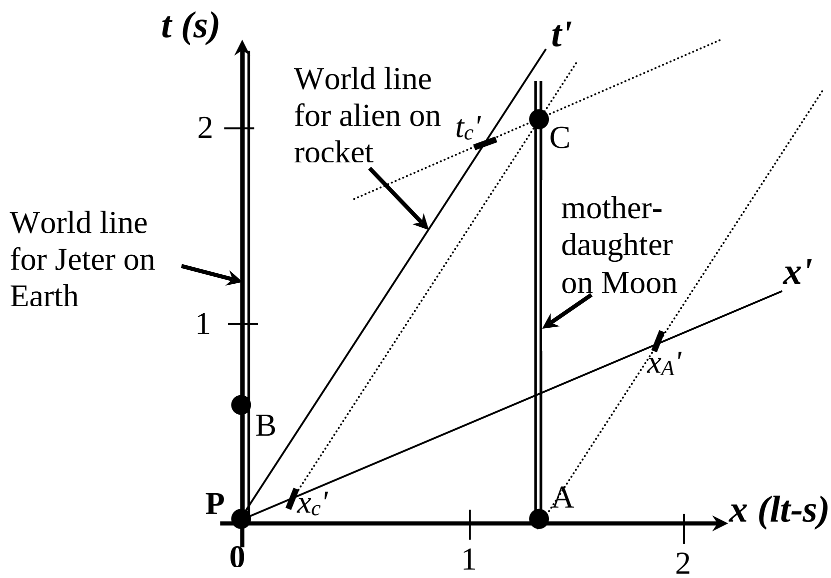


Figure 2.3 Extension of spacetime diagram in Example 3.

Special relativity helps us deal with the following question: how does an observer moving in a different reference view these same events? We won't worry here about the actual *numerical* values of x' and t' (the position and time as measured by a different observer), but we can say quite a lot about the *ordering* of events in space and time by looking at the spacetime diagrams.

We have added another world line to Figure 2.3, namely, the world line for a hypothetical alien whizzing past the Earth just as the pitch is thrown. This alien is monitoring the game to try to understand human culture. We assume the alien is traveling at a speed $0.5c$; hence, the world line has a slope of 2.

We have already commented that the world line of an observer in a primed frame is simply the t' axis for that frame, so we have labeled the alien's world line t'. But where should we put the x'-axis and what scale should we put on it? It turns out that to satisfy the invariance of the speed of light, we must draw the x'-axis at the same angle relative to the x-axis as the angle of the t'-axis relative to the t-axis. This means the slope of the x'-axis is equal to the speed v of the primed frame relative to the unprimed frame.

Recall that the t'-axis represents points where $x' = 0$. It turns out that x' is constant along any line parallel to the t'-axis. In other words, lines parallel to the t'-axis are *equal location* lines for the primed frame of reference, just as the t-axis and all lines parallel to it are each lines of equal location for the unprimed frame of reference. The same ideas work for events on lines parallel to the x or x' axes → events on a line parallel to the x-axis are simultaneous in the unprimed frame, and events on a line parallel to the x'-axis are simultaneous in the primed reference frame.

We can use these ideas to "read off" coordinates for events in both reference frames. As an example, let's look at event C in Figure 4.3. We have already commented that in the unprimed frame, its x location is 1.3 ls and its time is 2 s. The coordinates of this event in the alien's reference frame are determined by drawing lines parallel to the x' and t' axes (shown as dotted lines in Figure 4.3). The intersections of these construction lines with the opposing primed axis gives the x'_C and t'_C coordinates. The rules for determining coordinates can be summarized as follows:

To find x_C,	draw a straight line through C parallel to the t-axis and read off where it crosses the x-axis.
To find t_C,	draw a straight line through C parallel to the x-axis and read off where it crosses the t-axis.
To find x'_C,	draw a straight line through C parallel to the t'-axis and read off where it crosses the x'-axis.
To find t'_C,	draw a straight line through C parallel to the x'-axis and read off where it crosses the t'-axis.

Using this type of construction, we can see that although events A and C occur at the same place in the unprimed (Earth-Moon) reference frame, event C happens to the left of the event A in the primed (alien) reference frame. This is easy to understand: the alien is far from the Moon when event A happens, so A is far "to the right", whereas the alien is close to the Moon when event C happens, so from the alien's perspective, C isn't so far to the right, i.e., smaller x' coordinate.

But what about the ordering of events in *time*? We have commented that the invariance of the spacetime interval says that if two observers disagree about distances, then they will have to disagree about time intervals as well.

Think About This #2.4:
Look at the t' coordinates for events P and A. In the Earth-Moon reference frame, these events are simultaneous. What about in the alien reference frame?

We have said that any two events on a line parallel to the x'-axis are simultaneous in the primed frame of reference. Similarly two events that lie on a line parallel to the x-axis are simultaneous in the unprimed frame. However two different events cannot lie both on a line parallel to the x-axis and parallel to the x'-axis. Thus two events that are simultaneous in one frame cannot be simultaneous in the other frame. We explore this idea in the following example.

Example 4. Simultaneity is Relative. Einstein showed, with the following thought experiment, that two events which occur at the same time but at different places in one frame, occur at different times in another frame.

Imagine a train moving past a station. By chance, lightning happens to strike the front and back of the train at the same time according to observers on the station platform. Light pulses from these strikes travel toward the middle of the train, where a passenger observes their times of arrival. Do the light pulses arrive simultaneously or does one arrive before the other, and if so, which one?

Solution : Use a spacetime diagram, Figure 2.4, with the station at rest in the unprimed frame and the train at rest in the primed frame. The x- and x'-axes both lie along the track. The world line for the middle of the station is shown as the t-axis.

Because all parts of the train are at rest in the primed frame, we draw the world lines for the front and the rear ends of the train parallel to the t'-axis. Also, in Figure 2.4, we have chosen the world line for the passenger riding in the exact middle of the train to be the t'-axis. In the primed frame the front and rear world lines are then equidistant from the passenger, by definition.

The lightning strikes occur at points R and F on the world lines of the rear and front of the train. Because each strike represents an event and because these two events occur simultaneously in the station frame, R and F must be drawn on the same horizontal line. We arbitrarily choose this line to be at $t = 0$.

The light pulses produced by the lightning strikes travel with speed c from the event F back toward the passenger and from R forward toward the passenger. The pulse from F is represented by a world line of slope –1 and the pulse from R is represented by a world line of slope +1. Figure 2.4 shows that the pulse from F arrives at the passenger's world line (at F′) earlier (i.e. smaller value of t') than does the pulse from R, which arrives at R′.

The passenger must conclude that the front strike occurred before the rear strike because she is sitting in the middle of the train, equidistant from R and F, *and she knows the light pulses must have taken the same time (in her frame) to reach her*. By the same argument, an observer on the station platform who was at the exact middle of the train at $t = 0$ when the strikes occurred, sees the pulses at the same time. This is shown on the spacetime diagram by the fact that the world lines of the pulses cross the world line of the middle of the station at $x = 0$ (event M) at the same time.

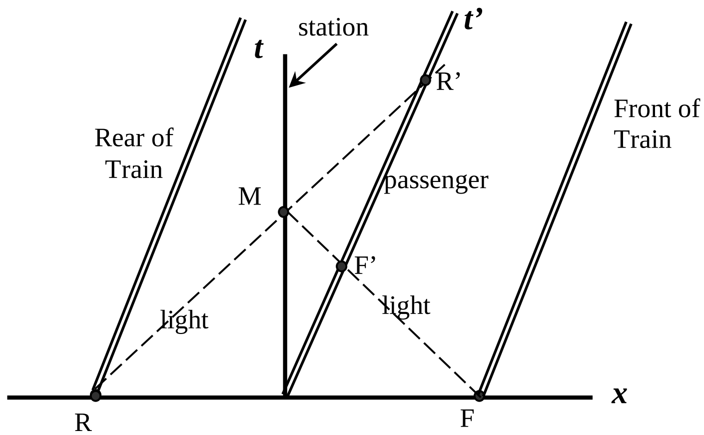
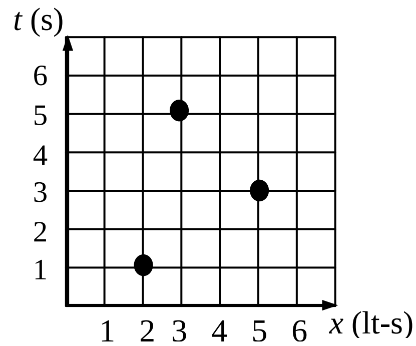


Figure 2.4 Spacetime diagram for a train moving at relative velocity of 0.6*c*. (World lines of light shown as dashed lines.)

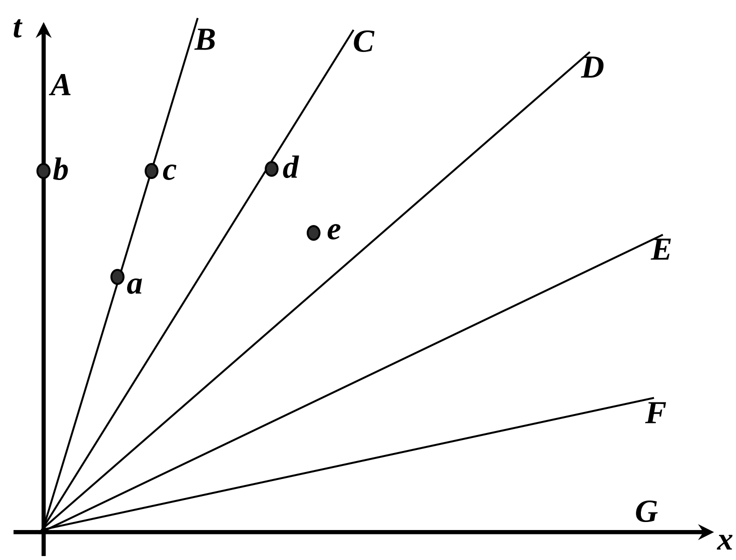
PROBLEMS

1) Three events are shown on the spacetime diagram. Event A occurs at 2 lt-s and 1 s, Event B occurs at 5 lt-s and 3 s, and Event C occurs at 3 lt-s and 5 s.



   a) Label the events A, B, and C.
   b) Calculate the value of the squared spacetime interval for each pair of events, i.e., find $(\Delta S_{AB})^2$, $(\Delta S_{AC})^2$, and $(\Delta S_{BC})^2$.
   c) Identify each interval as time-like, space-like, or light-like.
   d) In the frame shown, event A occurs before B, which occurs before C. Which pairs of events could have their time-order reversed (switching before and after) by choosing an appropriate reference frame?
   e) In the frame shown, event B occurs to the right of C, which occurs to the right of A. Which pairs of events could have their space-order reversed (switching left and right) by choosing an appropriate reference frame?
   f) Which events could be a "cause" for which other events?

2) The figure shows a spacetime diagram with seven straight lines through the origin labeled with capital letters A through G. Various events are marked as points with small letters *a* through *e*. The *x*-*t* axes belong to the Earth's reference frame.



   a) Which line is the world line of an object at rest relative to the Earth?
   b) Which line is the world line of a spaceship traveling at speed +0.3*c* relative to the Earth?
   c) Which line is a world line of a light pulse emitted by the spaceship as it passes the Earth?
   d) Which events happen simultaneously in the Earth frame?
   e) Which events happen simultaneously in the spaceship frame?
   f) Which pairs of events are clearly separated by space-like intervals? Which are clearly separated by time-like intervals?

3) In your reference frame, two firecrackers explode 4 lt-ns apart at the same time. In your friend's frame, the distance between the two events is determined to be 5 lt-ns. What is the time between those events in your friend's frame?

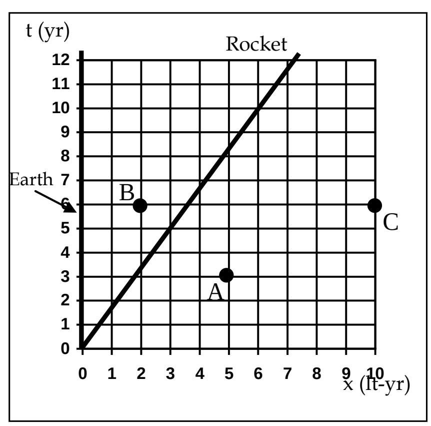
4) Jack lights and holds a match, and 60 seconds later, it goes out. Cheri, riding in a rocket past these events at constant speed, notes that, as measured in her frame, the match burned for 100 seconds.
   a) How far apart in Cheri's frame did these two events (lighting and going out) occur?
   b) As measured by Cheri, how far did the lit match travel, and how fast was it moving?
   c) As measured by Jack, how fast and how far did Cheri travel during the one minute the match was lit?

5) A train of rest length 40 lt-ns moves along the tracks at 0.8*c* and is struck by two lightning bolts. One bolt hits the front of the train and the other hits the back. According to track observers the bolts are simultaneous.
   a) How far apart on the tracks did the lightning bolts strike?
   b) According to riders on the train, how much time passed between the striking of the lightning bolts? Which occurred earlier?

6) A cosmic ray particle moving down toward Earth at speed 0.99*c* decays 2.00 microseconds after it was produced as measured in the rest frame of the particle.
   a) In the cosmic ray's rest frame the Earth is moving toward it. In this frame, how far, in light-μs, did the Earth travel during the particle's lifetime?
   b) Observers in the Earth's frame see the particle coming down toward Earth. How long did the particle live according to these observers and how far did it travel?

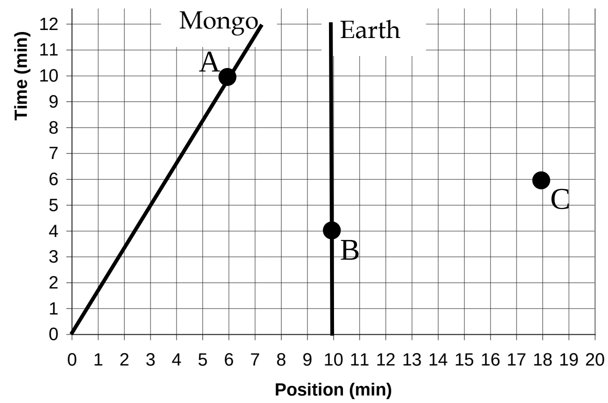
7) A spaceship crew wants to make the trip from Earth to Alpha Centauri (4 lt-yr apart in the Earth/Alpha Centauri rest frame) in only 3 years as measured by clocks on board their spaceship which travels at constant velocity. Determine how fast the spaceship must travel relative to Earth.

8) The spacetime diagram shows the worldlines of Earth and a rocket, as well as several labeled events.



   a) How fast is the rocket moving, relative to the Earth?
   b) Order events A, B, and C from earliest to latest in Earth's reference frame.
   c) Order events A, B, and C from earliest to latest in the Rocket's reference frame.
   d) Event B is the lighting of a signal beacon. Light leaves the beacon and travels toward the rocket. Also, light leaves the beacon and travels towards the earth. Draw and label the worldlines for each of these signals on the diagram.

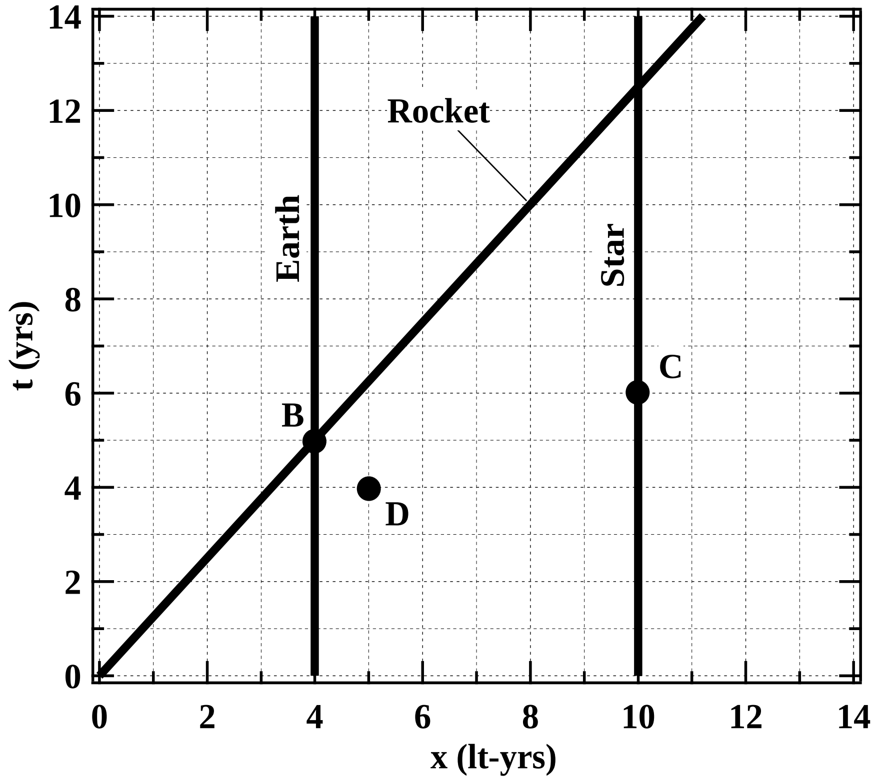
9) The spacetime diagram shows the wordlines of the planet Earth, the planet Mongo (on a collision course with the Earth), and several labeled events.
   a) Order events A, B, and C from earliest to latest in Mongo's frame.
   b) Event B is a Rocket passing by the Earth. In the Rocket Frame, events B and C are simultaneous. Draw and label the Rocket's worldline.
   c) Determine the speed of the Rocket, as measured by Earth observers.



10) A giant solar flare occurs on the Sun, which is located 8 lt-min from the Earth. Scientists on the Earth detect the light from the flare. At precisely the instant the scientists on Earth detect the solar flare light, a Klingon space ship passes by the Earth at speed $0.8c$, heading straight for the Sun.
   a) Construct a spacetime diagram for this situation. Label the following three events: A = Klingon ship hits Sun; B = flare occurs on Sun; C = Klingon ship passes Earth.
   b) Order the events A, B, C, from earliest to latest, according to Earth-based observers.
   c) Calculate the time intervals Δt between each pair of events (AB, AC, and BC), according to Earth observers.
   d) Calculate the intervals $\Delta t'_{BA}$ between events B and A, but now according to Klingon ship observers.
   e) Classify each of the intervals as space-like, time-like or light-like.

11) Farmer Brown, at rest in his (the primed) frame, carries what he measures to be a 5-meter long ladder through the front door of his barn. According to observers at rest with respect to the barn, Farmer Brown and his ladder are moving at a speed $0.80c$ towards the barn (alternately, Farmer Brown sees the barn moving towards him at speed $0.80c$.) In the barn's frame (the unprimed frame), the front door of the barn is at $x = 0$ and the back door is at $x = 4.0$ m. The front door closes at $t = 0$, just as the back end of the ladder passes through.
   a) In the barn frame, calculate the position of the front end of the ladder at the time the back end passes through the front door.
   b) Sketch a spacetime diagram from the rest frame of the barn. Show the front of the barn at $x = 0$ and the back of the barn at $x = 4.0$, along with world lines for the front and back of the barn. Also show worldlines for the front and back of the ladder.
   c) Explain why, in the barn frame, the 5-meter long ladder fits inside the 4-meter wide barn.
   d) Does Farmer Brown think the ladder fits inside the barn? Answer by calculating the position of the back door of the barn, in Farmer Brown's frame.

12) The spacetime diagram shows the world lines of the Earth, a Star, and a Rocket, as well as several labeled events.



a) On the diagram, label as "A" the event "Rocket passes Star."
b) Determine the speed of the Rocket, as measured by Earth observers.
c) Determine the time between passing the Earth and Passing the Star, as measured by Rocket observers.
d) Determine the distance between the Earth and the Star, as measured by Rocket observers.
e) Draw the world line of a lost satellite passing the Earth at the same time as the Rocket, but going away from the Star at a speed that is ½ of the Rocket speed (as determined by Earth observers.) Label this line "Satellite."
f) Determine the speed of the satellite as measured by Rocket observers.
g) Order the events A, B, C, D, from earliest to latest, as observed in the Earth-Star frame.
h) Order the events A, B, C, D, from earliest to latest, as observed in the Rocket frame.
i) In some reference frame, the events C & D are simultaneous. In that frame, what is the distance between events C & D?
j) Explain why no one could ever measure the proper time between events C & D.

Relativity 3: Relativistic Momentum and Energy

Learning Goals

1. **Know the modifications in the definitions of momentum and energy needed to maintain invariance of the conservation laws.**
2. **Given any two of a particle's dynamical quantities (*p, E, u, K*, and *m*) determine any of the others.**
3. **Show that $E^2-(pc)^2$ is an invariant quantity, related to the particle's mass and independent of velocity.**
4. **Calculate a particle's rest energy, and discuss its significance.**
5. **Specialize any of the equations relating *p, E, u*, and *K* to zero-mass particles.**

So far in our discussions of relativity, we have taken a very powerful principle – the *Principle of Relativity*, which states that the laws of physics are the same for observers in any inertial reference frame – and have used this principle to change completely our notions of how time and space work. But we are not yet done looking at the implications of this principle. It will be necessary to generalize the classical relations for energy and momentum to account for the strange behavior that we have already seen at relativistic velocities. And the new, relativistic equations for energy and momentum carry significant implications that change our notions of energy and matter. As we will see, we will also find a new invariant in this discussion; namely, the combination $E^2-(pc)^2$.

A. A note on units

When working with energy and momentum for small, subatomic particles (the ones that are most typically travelling at relativistic speeds), it is convenient to define a unit of energy called the "electron volt" (eV for short). One electron volt is the kinetic energy gained by an electron when accelerated through a 1 volt potential difference. Quantitatively, 1 eV = 1.6×10^{-19} J. An analogous energy unit might be a "superball-meter" – the amount of kinetic energy gained by a certain superball when dropped 1 m.

For high energy particles, the energies can get into the thousands, millions or billions of electron volts, so we also define 1 keV = 10^3 eV, 1 MeV = 10^6 eV, 1 GeV = 10^9 eV, etc.

Units for mass and momentum are also defined in terms of energy in relativity. For mass, we use eV/c^2 – "electron volts per c^2" – or keV/c^2, MeV/c^2, GeV/c^2. For momentum, we use eV/c (or keV/c, MeV/c, GeV/c). For example, an electron has a mass of 511 keV/c^2; conceptually, this means that an electron has a rest energy of 511 keV, or that its mass – if converted completely into energy – would produce 511 keV of energy.

Warning: when using these units, don't throw any numbers in for the c – it is part of the unit. So, the mass of an electron should be left as "511 keV/c^2" (or 0.511 MeV/c^2), ***not*** as $\cancel{511\ \text{keV}/(3.0\times10^8\,\text{m/s})^2}$ or $\cancel{511\ \text{keV}/(1\ \text{lt-s/s})^2}$.

B. New definitions for energy and momentum

You have learned previously that in interactions among low velocity particles in which the only forces are the interparticle forces (i.e. no *external* forces), the total momentum $\sum_i m_i \vec{u}_i$ and the total mass $\sum_i m_i$ are conserved. (As in Chapter 2, we use the symbol u to refer to the velocity of some particle as viewed from a reference frame, reserving v for the velocity of the reference frame itself.) For example, when particle 1 collides with particle 2 and particles 3, 4, and 5 emerge from the point of collision, we have used two conservation laws:

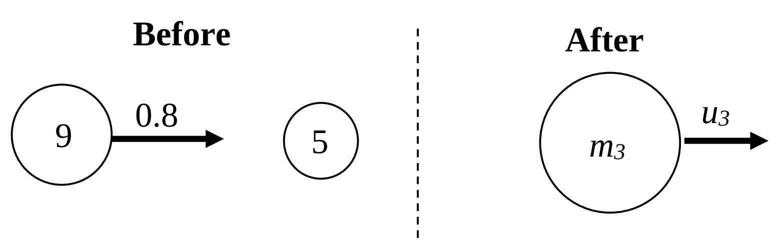
$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 + \vec{p}_5 \tag{3.1}$$

and

$$m_1 + m_2 = m_3 + m_4 + m_5 \quad \left(\begin{array}{c}\textbf{Caution : Valid in}\\ \textbf{classical regime only!}\end{array}\right) \tag{3.2}$$

After Einstein discovered the velocity transformation laws Eq. (1.3) and (1.4), he recognized that the classical definition of momentum ($\vec{p} = m\vec{u}$) was incompatible with Eq. (3.1) and the Relativity Principle. That is, for a given collision, classical momentum could be conserved in one frame but not another. An example will illustrate this:

Example 1. Say goodbye to the classical expression for momentum. (For this example, we use relativistic units of MeV for energy, MeV/c for momentum, MeV/c^2 for mass, and velocities in terms of c as described previously). The figure below shows a particle of mass 9 MeV/c^2 and speed $0.8c$ striking a stationary particle of mass 5 MeV/c^2, producing a single particle.



The classical laws Eqs. (3.1) and (3.2) would yield

(3.2): 9 MeV/c^2 + 5 MeV/c^2 = m_3
→ m_3 = 14 MeV/c^2

$$(3.1):\ (9\ \text{MeV}/c^2)(0.8c) + (5\ \text{MeV}/c^2)(0) = m_3u_3$$

$$\rightarrow u_3 = \frac{7.2\,\text{MeV}/c}{14\ \text{MeV}/c^2} = 0.514c$$

Transform now to a frame in which the final particle is at rest. This clearly means that we should view the collision from a spaceship traveling with particle 3 at 0.514c to the right, relative to the original observer.

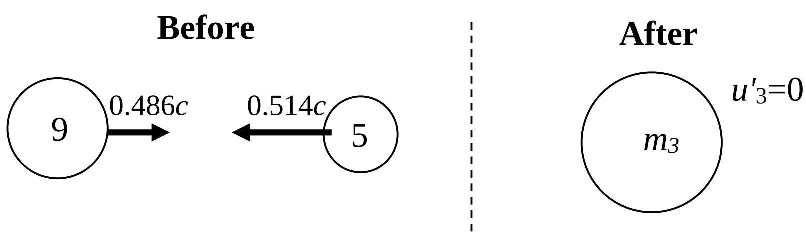
Eq. (2.4) gives

$$u'_3 = \frac{u_3 - v}{1 - u_3v/c^2} = \frac{0.514c - 0.514c}{1-(0.514)^2} = 0$$

$$u'_5 = \frac{u_5 - v}{1 - u_5v/c^2} = \frac{0 - 0.514c}{1-(0)(0.514)} = -0.514c$$

$$u'_9 = \frac{u_9 - v}{1 - u_9v/c^2} = \frac{0.8c - 0.514c}{1-(0.8)(0.514)} = 0.486c$$

So, in the new primed frame, the collision looks like this:



Checking the classical momentum conservation law in the new frame gives

$$(9\ \text{MeV}/c^2)(0.486\text{c}) + (5\ \text{MeV}/c^2)(-0.514c) = m_3(0)$$

But the left side of this equation here works out to be 1.80 MeV/c which is NOT equal to the right side (which is 0). So, classical momentum is ***not*** conserved in this new frame. Therefore, either (a) conservation of momentum isn't a valid law of physics; (b) the Relativity Principle (invariance of the laws of physics) is violated; or (c) we need a new definition for momentum.

You probably won't be surprised to hear that Einstein wasn't about to give up on the Relativity Principle because of this argument. After all, he had already redefined time and space to make the Principle work. And although the expression $m\vec{u}$ for momentum does not lead to invariance for high velocity collisions, there are attributes of particles involving their masses and velocities that do produce invariant conservation laws. These quantities are called relativistic momentum and relativistic energy, or more simply, momentum and energy. They are defined by

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \qquad \left(\begin{array}{c}\textbf{\textit{momentum in terms of}}\\ \textbf{\textit{mass and velocity}}\end{array}\right) \tag{3.3}$$

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \left(\begin{array}{c}\textbf{\textit{energy in terms of}}\\ \textbf{\textit{mass and velocity}}\end{array}\right) \tag{3.4}$$

Think About This #3.1:
If a particle is at rest, so that u = 0, what is its momentum and its energy according to Eqs (3.3) and (3.4)?

Think About This #3.2:
If a particle is moving faster and faster, so that u approaches c, what happens to its momentum and its energy according to Eqs (3.3) and (3.4)? For the calculus-track students, another way of asking this is to find $\lim\limits_{u\to c^-} \vec{p}$ and $\lim\limits_{u\to c^-} E$.

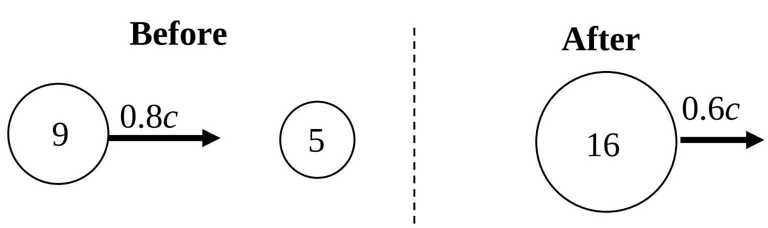
Einstein was motivated to define momentum and energy in this way because conservation of energy and momentum and energy defined in this new way are invariant, as we will show with an example below. Of course, motivation is all very nice, but the most compelling reason that the momentum and energy of a particle must be defined this way instead of in the classical way is because experiments with high-speed particles conserve these new relativistic quantities and not those given by the classical definitions.

Let's explore this invariance by redoing Example 1 using Einstein's new definitions and the conservation laws:

$$\vec{p}_{before} = \vec{p}_{after} \tag{3.5}$$

$$E_{before} = E_{after} \tag{3.6}$$

Example 2. The figure below shows a particle of mass 9 MeV/c² and speed 0.8c striking a stationary particle of mass 5 MeV/c², producing a single particle.



You might not be too happy here with the final particle having a mass of 16 MeV/c², but hold on a little longer – we'll explain this shortly. (A little preview – this might be a good time to take a pen and scribble Eq. (3.2) out of existence.) In the next chapter, we'll learn more rigorously how to determine the correct attributes of the final particle. Here we just want to check the conservation laws.

The relativistic definitions and conservation laws Eqs. (3.3) – (3.6) yield

$$\frac{(9\,\text{MeV/c}^2)(.8c)}{\sqrt{1-(.8)^2}}+0=\frac{(16\text{ MeV/c}^2)(.6c)}{\sqrt{1-(.6)^2}} \qquad \text{(momentum)}$$

$$\frac{(9\,\text{MeV/c}^2)\text{c}^2}{\sqrt{1-(.8)^2}}+\frac{(5\,\text{MeV/c}^2)\text{c}^2}{\sqrt{1-(0)^2}}=\frac{(16\text{MeV/c}^2)\text{c}^2}{\sqrt{1-(.6)^2}} \qquad \text{(energy)}$$

The momentum equation gives 12 MeV/c = 12 MeV/c, while the energy equation gives 15 MeV + 5 MeV = 20 MeV. So the conservation laws are satisfied in this frame.

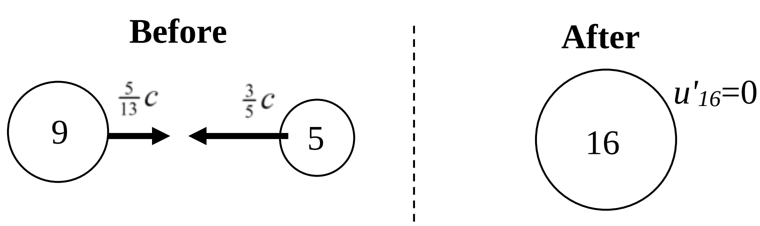
Now, transform to a frame in which the final particle is at rest, by viewing from a spaceship moving at 0.6c to the right. The velocity transformations give

$$u'_{16}=\frac{u_{16}-v}{1-u_{16}v/c^2}=\frac{0.6c-0.6c}{1-(0.6)^2}=0$$

$$u'_5=\frac{u_5-v}{1-u_5v/c^2}=\frac{0-0.6c}{1-(0)(0.6)}=-0.6c=-\frac{3}{5}c$$

$$u'_9=\frac{u_9-v}{1-u_9v/c^2}=\frac{0.8c-0.6c}{1-(0.8)(0.6)}=\frac{5}{13}c\approx 0.385c$$

In this new frame, the collision looks like this:



When we check the relativistic conservation laws in this new frame, we find:

$$\frac{(9\text{ MeV/c}^2)(5c/13)}{\sqrt{1-(\frac{5}{13})^2}}+\frac{(5\text{ MeV/c}^2)(-3c/5)}{\sqrt{1-(\frac{3}{5})^2}}=\frac{(16\text{ MeV}/c^2)(0)}{\sqrt{1-0^2}} \qquad \text{(momentum)}$$

$$\frac{(9\text{ MeV}/c^2)c^2}{\sqrt{1-(\frac{5}{13})^2}}+\frac{(5\text{ MeV}/c^2)c^2}{\sqrt{1-(\frac{3}{5})^2}}=\frac{(16\text{ MeV}/c^2)c^2}{\sqrt{1-0^2}} \qquad \text{(energy)}$$

The momentum equation gives

$$\frac{15}{4}\text{MeV}/c-\frac{15}{4}\text{MeV}/c=0$$

which checks out, while the energy equation gives

$$\frac{39}{4}\text{MeV}+\frac{25}{4}\text{MeV}=16\text{ MeV}$$

which also checks out. This means the conservation laws ***are*** true in both the original and the new frame, and the Relativity Principle is upheld with Einstein's new definitions.

This may be a nice argument on paper, but does it work in practice? Are relativistic momentum and energy, rather than classical momentum and mass, really conserved in particle interactions? The answer is an emphatic **YES**! In millions of interactions observed in high-energy particle accelerators, relativistic momentum and energy have always been found to be conserved to within experimental uncertainty.

C. Other useful relations between mass, velocity, momentum, and energy

We now have relativistic expressions for energy and momentum given by Eqs (3.3) and (3.4), which are in terms of mass and velocity. These expressions can be combined (you'll do this in a homework problem) to obtain:

$$E^2=(pc)^2+(mc^2)^2 \qquad \left(\begin{array}{c}\textbf{\textit{energy in terms of}}\\ \textbf{\textit{momentum and mass}}\end{array}\right) \tag{3.7}$$

which gives energy in terms of momentum and mass. As we'll see in the next chapter, this is actually the most useful of all the energy and momentum relations. It applies to ***every*** particle in ***every*** situation. (We'll see that Eqs. 3.3 and 3.4 aren't very useful for "particles" of light.) It is also convenient in that it doesn't have those square roots, which can be challenging to work with.

Similarly, beginning from Eqs (3.3) and (3.4), we obtain:

$$\vec{u} = \frac{\vec{p}\,c^2}{E} \qquad \left(\textbf{\textit{velocity in terms of momentum and energy}}\right) \tag{3.8}$$

which gives velocity in terms of momentum and energy. As with Eq (3.7), you will show how to derive this formula in a homework problem.

From equations (3.3), (3.4), (3.7), and (3.8), we can use any two of energy, momentum, speed, and mass to calculate the other two.

D. Another invariant

Eq. (3.7) can be rearranged to give

$$(mc^2)^2 = E^2 - (pc)^2 \Rightarrow m^2c^4 = E^2 - p^2c^2 \Rightarrow m^2 = E^2/c^4 - p^2c^2/c^4$$
$$\Rightarrow m^2 = (E/c^2)^2 - (p/c)^2$$

This expression: $m^2 = (E/c^2)^2 - (p/c)^2$ is an invariant quantity. We have encountered and worked with two other invariant quantities by this point: the speed of light c and the spacetime interval ΔS. Recall from chapter 2, we defined the square of the interval as

$$(\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2 \tag{2.1}$$

As stated earlier, $(\Delta S)^2$ is an invariant – observers in different reference frames will agree on the value of this interval for any two events:

$$\begin{array}{ccc} (\Delta S)^2 & = & (\Delta S')^2 \\ (c\Delta t)^2 - (\Delta x)^2 & = & (c\Delta t')^2 - (\Delta x')^2 \end{array} \tag{2.2}$$

where Δx and Δt are the distance and time between two events as measured in one reference frame, and $\Delta x'$ and $\Delta t'$ are the distance and time between the same two events as measured in another reference frame.

Compare Eq. (2.1) with

$$m^2 = (E/c^2)^2 - (p/c)^2 \tag{3.8}$$

Given any object or particle with energy E and momentum p as measured by an observer in a reference frame, this observer can easily calculate the value of m^2 (or m) for that object.

If a different observer is in a primed reference frame and determines E' and p' for the same particle, she will find that if she calculates

$$(m')^2 = (E'/c^2)^2 - (p'/c)^2$$

then she will get exactly the same value for m' that the first observer found for m. In other words

$$\boxed{\begin{array}{ccc} (m)^2 & = & (m')^2 \\ (E/c^2)^2 - (p/c)^2 & = & (E'/c^2)^2 - (p'/c)^2 \end{array}} \tag{3.9}$$

In the same manner that we used the invariant spacetime interval to relate Δx and Δt as measured in one reference frame to $\Delta x'$ and $\Delta t'$ in another reference frame, we can use the invariance of m to relate E and p in one frame to E' and p' in a different frame.

What *is* this invariant m? This is simply the mass of the object. So, in words, the invariance expressed in Eq (3.9) states that all observers agree about the mass of an object.[1]

We can re-write the expression given in Eq (3.9) in a slightly more convenient form:

$$\boxed{\begin{array}{ccc} (mc^2)^2 & = & (m'c^2)^2 \\ E^2 - (pc)^2 & = & (E')^2 - (p'c)^2 \end{array}} \tag{3.10}$$

Think About This #3.3:
What are 3 invariant quantities we have encountered so far? What is important about invariant quantities –or- why are invariant quantities so useful?

E. Rest Energy and Kinetic Energy

Let's look more closely at what we called the relativistic energy of a particle in Eqs (3.4) or (3.7). If the particle is at rest, then u = 0 and p = 0. Then, the formulas for energy reduce to $E = mc^2$, perhaps the most famous formula in all of physics. So we see that a particle has energy even when it's not moving!

---

[1] You may hear people saying that in relativity, "a person's mass increases as he approaches the speed of light." This is an unfortunate claim. What they are doing is saying, "Well, since $p = mu/\sqrt{1-u^2/c^2}$, we're going to call $m/\sqrt{1-u^2/c^2}$ the relativistic mass so that we can hold on to the $p=mu$ definition of momentum." There is no compelling reason to do this – there is nothing in relativity that requires us to redefine mass as opposed to redefining momentum (which is what we have done here).

This energy is called the *rest energy*, E_0. That is:

$$E_0 = mc^2 \tag{3.11}$$

This was a remarkable result – the implications are two-fold:

(a) first, the relation implies that matter and energy aren't separate quantities, but are really just different forms of the same thing; and

(b) second, implied in this relation is the possibility of converting between matter and energy. *And the conversion factor is c^2 – a **huge** number (when expressed in "everyday" units of m^2/s^2 or J/kg)!* To get an idea of the magnitude of this factor, try computing the amount of energy contained in a 1 g paperclip.

Think About This #3.4:
What is the (rest) energy contained in a 1 g (0.001 kg) paperclip? Express your answer in Joules, which means using 3.0 x 10^8 m/s for c. For comparison, consider that you would have to lift this paperclip about 100 m (about 330 feet) to increase its gravitational potential energy by 1 J, or that this paperclip would need to travel at about 45 m/s (about 100 mph) to have a kinetic energy of 1 J. What are your thoughts about the energy contained in this paperclip?

We have called the energy associated with a particle's motion its kinetic energy. However in the relativistic regime, kinetic energy is not expressed as $K = \frac{1}{2}mv^2$. Instead, it is defined as the difference between a particle's energy when it is moving and its rest energy,

$$K = E - mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 \ . \tag{3.12}$$

However, this doesn't resemble the classical expression for kinetic energy, which as you recall was $K = \frac{1}{2}mv^2$ (or, since we use v for the velocity of the frame and u as the velocity of the particle, $K = \frac{1}{2}mu^2$). Since we know that the classical expression for kinetic energy is valid in the low speed regime, the relativistic kinetic energy given by Eq. (3.12) must somehow reduce to the classical expression when the speed of the particle is small compared to the speed of light. We show the connection between the relativistic and classical forms of kinetic energy in the next example.

Example 3. Classical expression for kinetic energy. Use the binomial approximation in Eq (3.12) to find an approximate expression for K when u is much smaller than c.

Solution: The binomial expansion states that $(1-\varepsilon)^{-1/2} \approx 1+\frac{1}{2}\varepsilon+\ldots$ if ε is small.

We write $\frac{1}{\sqrt{1-u^2/c^2}}$ as $\left(1-u^2/c^2\right)^{-1/2}$. Then, Eq (3.12) becomes

$$\begin{aligned} K &= mc^2\left(1-u^2/c^2\right)^{-1/2} - mc^2 \\ &\approx mc^2\left(1+\tfrac{1}{2}u^2/c^2+\cdots\right) - mc^2 \\ &\approx \tfrac{1}{2}mu^2 \end{aligned} \tag{3.12}$$

Thus we see that the classical expression for kinetic energy is only a low-velocity approximation to the correct expression, given by Eq. (3.11).

F. Photons: Particles with Zero Mass

How do we deal with the energy and momentum of light? In the same year that Einstein published his first paper on Special Relativity, he also proposed that light must be considered to be composed of particles which are now called *photons*. (This was the first of the three 1905 papers discussed at the beginning of Chapter 1.) Since light always travels at a speed c in a vacuum, then photons in a vacuum must travel at that speed regardless of the reference frame of the observer. But if we look back at Eqs. (3.3) and (3.4), we find that the denominators of both equations are *zero* for a particle moving at the speed of light, which is clearly problematic, as we can't divide by zero.

The way to resolve this dilemma is to postulate that photons are particles with *zero* mass (m = 0). Eqs (3.3) and (3.4) are still not very useful in this case (a fraction which has zero in both the numerator and denominator is indeterminate). However, Eq (3.7) is useful; for m = 0, it becomes

$$E = |pc| \quad \textit{for massless particles } \boldsymbol{only} \tag{3.13}$$

Note that while Eq (3.7) is valid for all particles, Eq. (3.13) is valid ***only*** for massless particles.

G. More experimental evidence

Now that we have introduced the relativistic relations for energy and momentum, we can discuss some additional pieces of evidence that Einstein's theory of relativity is, in fact, correct. The following examples can be added to those presented in Chapter 1. Remember that if even *one* of these experiments had disagreed with Einstein's theory, then the *entire* theory would have to be thrown out since everything is internally consistent.

- **Particle accelerators**: As we already discussed in Chapter 1, subatomic particles are frequently accelerated in high energy experiments to speeds *very close* to c, but no one has ever managed to accelerate a particle with mass to a speed greater than c. There's more here, though: as the particle's speed (relative to the laboratory) gets closer and closer to c, the amount of energy that has to be added to increase the speed further gets larger and larger, diverging as the speed approach c. So, for instance, the amount of energy that needs to be added to accelerate a particle from $0.98c$ to $0.99c$ has been found experimentally to be much larger than the energy to accelerate the same particle from $0.97c$ to $0.98c$, and in fact, much larger than that predicted classically. As is the case with all other tests of relativity, the amount of energy to be added agrees perfectly with Einstein's predictions, to within experimental uncertainty.

- **Collisions of high-energy particles**. When subatomic particles are slammed into each other with high energies, new particles are actually *created* that weren't there before the collision. These collisions are converting kinetic energy (KE) into matter, and this is done all the time in particle accelerators. (This is, in fact, the main tool that physicists use to study massive subatomic particles.) This is an experimental result that simply cannot be explained classically. Once again, though, the results agree perfectly with Einstein's theory. We will be discussing this in more detail in the next chapter, and some of you might know about particle accelerators, where much exciting new physics is discovered.

- **Matter-to-KE conversions**. One of the most convincing and most dramatic tests of Einstein's theory of relativity occurred on July 16, 1945, in New Mexico when the first atomic bomb was exploded, converting matter into a horrifying amount of kinetic energy (don't forget that factor of c^2 in the famous $E = mc^2$ equation). Since then, there have been quite a few additional such demonstrations of Einstein's theory. (And again, the quantitative aspects of these demonstrations agree perfectly with the theory.)

It isn't necessary to explode a bomb to convert matter into energy. Nuclear energy has found peaceful (if controversial) applications in the area of power generation. We will discuss nuclear power generation more in the next chapter (including *fusion* power – still being developed – which doesn't produce any long-lasting radioactive waste).

The various formulas introduced in this chapter are summarized in the following table:

Table 3.1 Formulas relating *p, E, u, K,* and *m*

$\vec{p} = \dfrac{m\vec{u}}{\sqrt{1-u^2/c^2}}$	$\left(\textit{momentum in terms of mass and velocity}\right)$	(3.3)
$E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}}$	$\left(\textit{energy in terms of mass and velocity}\right)$	(3.4)
$E^2 = (pc)^2 + (mc^2)^2$	$\left(\textit{energy in terms of momentum and mass}\right)$	(3.7)
$\vec{u} = \dfrac{\vec{p}c^2}{E}$	$\left(\textit{velocity in terms of momentum and energy}\right)$	(3.8)
$K = E - mc^2$	$\left(\textit{kinetic energy in terms of energy and mass}\right)$	(3.12)
$E = pc$	$\left(\textit{energy in terms of momentum for zero mass particle}\right)$	(3.13)

We also summarize the various invariant quantities we have encountered so far in our study of special relativity:

speed of light: c

spacetime interval:

$$\boxed{\begin{array}{ccc} (\Delta S)^2 & = & (\Delta S')^2 \\ (c\Delta t)^2 - (\Delta x)^2 & = & (c\Delta t')^2 - (\Delta x')^2 \end{array}}$$

mass:

$$\boxed{\begin{array}{ccc} (mc^2)^2 & = & (m'c^2)^2 \\ E^2 - (pc)^2 & = & (E')^2 - (p'c)^2 \end{array}}$$

PROBLEMS

1) A particle of mass $3m$, moving at speed $0.60c$ in the $+x$-direction, collides with and sticks to a particle of mass $2m$ originally at rest. The single composite particle also moves to the right after the collision.
   a) Calculate the initial total momentum before impact, using the classical definition, $p = mu$, for momentum.
   b) Assuming conservation of mass as well as classical momentum, find the velocity of the composite particle of mass $5m$ after the collision.
   c) Now transform to a primed frame in which the particle of mass $3m$ is at rest. Use the relativistic velocity transformation to compute the velocities u'_{3m}, u'_{2m}, and u'_{5m} in the primed frame.
   d) Show that in the primed frame, the (classical) momentum mu' is not conserved by computing the total momentum before the collision and the total momentum after the collision.

2) An electron ($m_{electron}$ = 0.511 MeV/c^2 = 511 KeV/c^2) is accelerated from rest up to a velocity $u = 0.99c$. Note that this is straightforward to achieve even in an out-dated particle accelerator, so the calculations below give a clear way to test between classical mechanics and special relativity.
   a) Using the classical definition of kinetic energy, $K = \frac{1}{2}mv^2$, calculate the change in the electron's kinetic energy in units of KeV. This change in kinetic energy has to be added to the electron in the form of work done by the accelerator.
   b) Using the relativistic definition of kinetic energy, calculate the change in the electron's kinetic energy in units of KeV. This change in kinetic energy has to be added to the electron in the form of work done by the accelerator.

3) Electron A has a total energy of 1.0 MeV. Electron B has a kinetic energy of 0.25 MeV. Electron C has a kinetic energy of 0.75 MeV. Electron D has a momentum of 1.0 MeV/c. For each of the electrons A through D, determine its energy, momentum, kinetic energy, and speed.

4) A certain particle has a total energy of 1.20 MeV and a momentum of 0.95 MeV/c. Calculate the particle's mass, kinetic energy, and velocity.

5) Compute the momentum and velocity of a proton (m_{proton} = 938 MeV/c^2) that has a total energy equal to 7 times its rest energy.

6) A proton (mass 938 MeV/c^2) is traveling at velocity $0.60c$ relative to a spaceship which itself is traveling at velocity $0.80c$ relative to Earth. Calculate the velocity and then the energy and momentum of the proton as measured in the Earth frame.

7) A particle's energy and momentum in one frame are 41 MeV and 40 MeV/c respectively. Find the particle's energy and momentum in a primed frame in which the particle's speed is $u' = 0.8c$.

8) Given a particle with $E' = 21$ MeV and $p' = 15$ MeV/c in the primed frame, and $E = 20$ MeV in the unprimed frame, determine the mass m and momentum p of the particle.

9) A certain J-boson has mass of 150 MeV/c^2, speed of $0.8c$, and, and total energy of 250 MeV. Determine the J-boson's momentum and kinetic energy.

10) A proton (mass 938 MeV/c^2) has kinetic energy of 1.2 GeV. Determine this proton's momentum and speed.

11) Combine Eqs. (3.3) and (3.4) to derive Eq. (3.7).

12) Combine Eqs. (3.3) and (3.4) to derive Eq. (3.8).

13) Show, from Eqs. (3.7) and (3.8) that any massless particle moves at the speed of light and that if a particle moves at the speed of light it must have zero mass.

Relativity 4: Application of the Relativistic Conservation Laws

Learning Goals

1. **Apply the relativistic conservation laws for momentum and energy to decays or "explosions", in which one particle decays into two (or more) particles moving along a straight line, including cases in which some or all the outgoing particles have zero rest mass.**
2. **Apply the relativistic conservation laws for momentum and energy for simple collisions with all particles traveling along a line.**
3. **Describe the processes of nuclear fusion and fission, and explain how these processes result in energy production.**
4. **Given information about nuclear masses, calculate the amount of kinetic energy gained in a fusion or fission process.**

You should now understand why Einstein's first postulate, that the laws of physics are the same in different inertial reference frames, requires new definitions of momentum and energy. The classical momentum is not conserved, nor in general is the total mass of the particles in an interaction. In place of these, relativistic momentum and relativistic energy are conserved, and they are conserved in any inertial frame. Here we investigate several examples of particle interactions in which we apply these new conservation laws.

A. Changes of Rest Energy

Much of the light you see comes from changes in rest energy of atoms. Examples are sunlight, light from a candle flame, a lightning flash, light emitted by a fluorescent lamp, light from the phosphor coating on the screen of a television set or a video monitor, and laser light. In all these examples, the basic mechanism is that an atom in an "excited" state releases its energy in the form of a photon, with the atom going into its ground (lowest possible) state, or into an excited state of lower energy. We can represent the emission process by the simple reaction equation

$$A^* \rightarrow A + \gamma \tag{4.1}$$

Here **A*** represents the excited atom, **A** the atom in its ground or lowest state, and γ (Greek *gamma*) the photon. This reaction is illustrated in Figure 4.1.

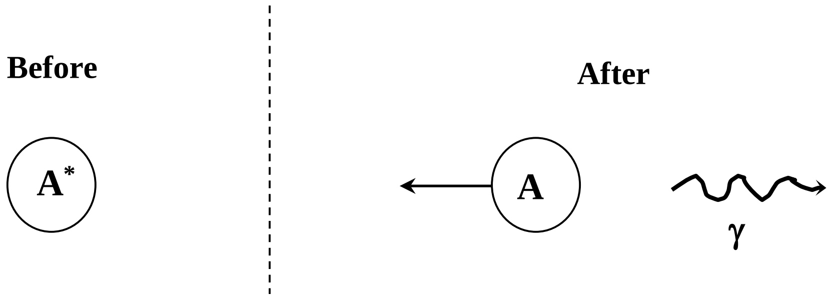


Figure 4.1. An excited atom emits a photon and recoils.

In Figure 4.1, the excited atom is shown at rest, so all of its energy is rest energy and it has no momentum. But the photon has energy, and from the relation $E = pc$, it also has momentum. And because momentum must be conserved, the atom recoils. We can write the conservation of energy equation for the reaction in Eq. (4.1) as follows

$$\text{rest energy of A}^* = \text{rest energy of A} + \text{kinetic energy of A} + \text{energy of } \gamma \tag{4.2}$$

Because both the kinetic energy of A and the photon energy are positive numbers, the rest energy (i.e., the mass) of the excited-state atom must be greater than that of the ground-state atom. Therefore, in the emission process rest energy, i.e., mass, is converted to kinetic energy.

When light is absorbed by an atom, exactly the opposite effect occurs. The atom begins in its ground state, absorbs the photon energy and goes into an excited state. Again, by conservation of energy, the excited atom must have more rest energy than the ground-state atom

Another everyday example of changing rest energy occurs in chemical reactions. For example, the reaction for the oxidation of a carbon atom

$$C + O_2 \rightarrow CO_2 \tag{4.3}$$

is known to release energy in the form of one or more photons. Therefore the sum of the masses of C and O_2 must be greater than the mass of the carbon dioxide molecule. The change in rest energy in the case of chemical reactions is typically on the order of 1 eV (or 1.6×10^{-19}J). Much larger energies, on the order of 1 MeV, are involved in nuclear reactions. An example of a nuclear reaction is the decay of a neutron into a proton, an electron, and an *electron antineutrino*:

$$n \rightarrow p + e^- + \overline{\nu}_e \tag{4.4}$$

Here the excess mass of the neutron over the mass of the proton plus electron (the electron antineutrino has *very* small mass) is converted to the kinetic energy of the three reaction products.

Another important example of changes in rest mass is the production of new particles in a high energy particle accelerators. In these accelerators, high-speed particles are shot at target particles and some of the kinetic energy of the incoming particles is converted to rest energy. In this way hundreds of new particles, most with lifetimes between 10^{-10} and 10^{-23} s, have been produced.

B. General Strategy for Applying the Relativistic Conservation Laws

In a typical problem you are given information about the particles before an interaction and asked to compute certain properties of the outgoing particles after the interaction. You do this by writing down equations that express the fact that the sum of the incoming momenta is equal to the sum of the outgoing momenta and the sum of the incoming energies is equal to the sum of the outgoing energies. What quantities should be used in writing these equations? Here is some time-saving advice.

First rule:

> *Always write the conservation of momentum and conservation of energy equations in terms of momentum and energy or mass variables, never in terms of velocity or kinetic energy.*

This rule keeps the algebra as simple as possible – it gets around having to solve simultaneous equations with the $\sqrt{1-v^2/c^2}$ terms that can make the algebra messy. For example, if you are given the velocity of one or more particles in the problem statement, first calculate the momentum and energy of each particle from the given velocities.

Second rule:

> *When working with "*eV*" units (e.g.,* MeV *for energy,* MeV/c *for momentum,* MeV/c^2 *for mass), don't ever put any numbers in for the speed of light c. Just leave it as "c". The units will then automatically take care of themselves*

For example, if you have a motionless electron, its energy can be obtained from $E^2 = (pc)^2 + (mc^2)^2$. Since the electron is motionless, its momentum $p = 0$, so its energy is just its rest energy $E = mc^2$ = (0.511 MeV/c^2) * c^2 = 0.511 MeV. Hopefully this is familiar to you after your previous work, especially in Chapter 3. If needed, review Chapter 3, section A.

We will work out an example similar to that in Eq (4.1) and Figure 4.1, but this time with numbers.

Example 1. **Emission of a photon by a nucleus.**
An excited atomic nucleus, of mass 5.00 GeV/c² and at rest, as in figure 4.2, decays to its ground state by emitting a photon of energy 2.00 GeV. Calculate the recoil velocity and mass of the ground-state nucleus.

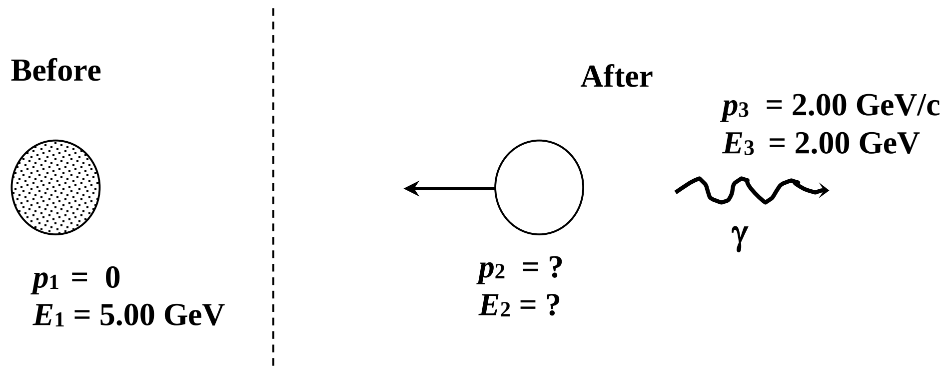


Figure 4.2 Emission of a photon by a nucleus

Solution: First *draw a picture, and label each particle with its value of energy and momentum.* Before the decay the excited nucleus has zero momentum because it is at rest. And from $E^2 = (pc)^2 + (mc^2)^2$, with $p = 0$, we know its energy is the same as its rest energy, namely 5.00 GeV.

After the decay the ground-state nucleus recoils with unknown energy and momentum, E_2 and p_2. Also, the emitted photon has an energy of 2.00 GeV, as specified in the problem. And because the photon's mass is zero its momentum has the same numerical value as its energy. Notice that in the diagram there are two unknowns, the energy and momentum of the recoiling ground-state nucleus. We plan to solve for these two unknowns with two equations, the energy and momentum conservation equations.

Looking at the diagram, we write down the energy conservation equation in terms of the symbols and numerical quantities shown in the diagram:

$$5.00 \text{ GeV} = E_2 + 2.00 \text{ GeV}.$$

Similarly, we write the momentum conservation equation in terms of symbols and numerical quantities shown in the diagram:

$$0 = p_2 + 2.00 \text{ GeV/c}$$

From these conservation-law equations we easily solve for the energy and momentum of the recoiling nucleus to obtain $E_2 = 3.00$ GeV and p_2 = –2.00 GeV/c. Now that we've obtained expressions for the energy and momentum of the recoiling ground-state nucleus, we can find its velocity most directly using Eq. (3.8).

$$u_2 = \frac{p_2 c^2}{E_2} = \frac{(-2.00 \text{ GeV}/c) * c^2}{3.00 \text{ GeV}} = -0.67\ c,$$

and its mass from

$$\begin{aligned} m_2 c^2 &= \sqrt{E_2^2 - p_2^2 c^2} \\ &= \sqrt{(3.00 \text{ GeV})^2 - (2.00 \text{ GeV}/c)^2 * c^2} \\ &= 2.24 \text{ GeV} \end{aligned}$$

so the mass m_2 = 2.24 GeV/c^2.

Notice that even though we were asked to find the velocity and mass of the recoiling nucleus, we didn't use these variables in our analysis until the very end, after we solved for its energy and momentum.

The example just shown was for an emission of a photon from an initially stationary excited-state nucleus. This can be considered as a decay, or an explosion: a single particle before the interaction, and multiple particles after. We can also use the same general problem solving strategy and mathematical tools to consider the case of collisions between particles.

C. Nuclear masses, fusion and fission

A particularly important application of the relativistic conservation laws is nuclear power generation. There are two main approaches – fusion and fission. Nuclear fusion involves the merging (fusing) of two light nucleii (usually Hydrogen) to form a more massive nucleus (usually Helium), whereas fission involves the splitting of a very massive nucleus (e.g., Uranium) into two or more lighter nucleii. For the process to release kinetic energy, conservation of relativistic energy requires that the end product(s) have a smaller total mass than the initial nucleus or nucleii.

Figure 4.3 shows a plot of the masses of the elements, divided by the total number of protons and neutrons (nucleons) in the nucleus of each atom.

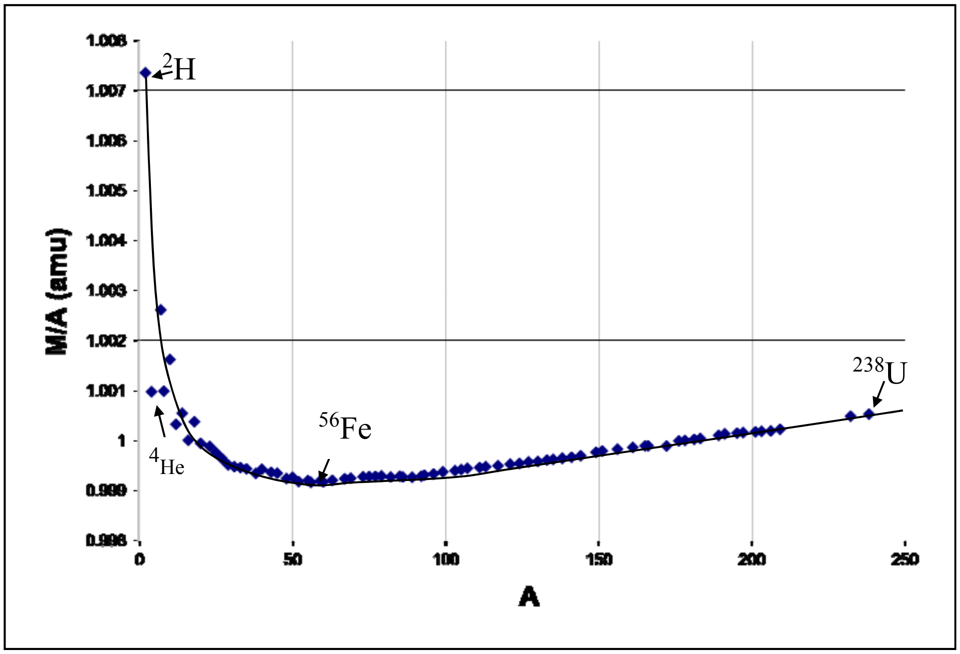


Figure 4.3. Plot of average mass per nucleon (protons and neutrons) for the elements versus the number of nucleons A in the atom. The vertical axis has units amu which stand for atomic mass unit.

This plot is useful when considering fusion and fission processes. The fusion of two ^{2}H nucleii to form a single ^{4}He nucleus results in a lower overall mass; consequently, this process "releases" kinetic energy. Let's examine this reaction in more detail.

Example 2. Fusion reaction: $^2\text{H}+^2\text{H}\rightarrow^4\text{He}$ **.**

Our strategy will be to calculate the mass of the reacting particles before the reaction and the mass of the produced particles after the reaction. This will tell us the (rest) energy before and the (rest) energy after.

- If there is more rest energy *before* the reaction than after the reaction, we know by conservation of energy that some of the mass must have been converted to kinetic energy. This kinetic energy shows either in the particles resulting from the reaction moving or through the creation of photons (which in some sense *are* kinetic energy). This is net production of kinetic energy, which can be used for example to heat water in a turbine generator to produce electricity.
- If there is more rest energy *after* the reaction than before, we know by conservation of energy that some (kinetic) energy must have been converted to mass. This requires an absorption of kinetic energy and is not useful for power generation.

In this reaction, two $^{2}\mathrm{H}$ nuclei (these are actually isotopes of hydrogen called deuterium) fuse together, resulting in the formation of $^{4}\mathrm{He}$ (an isotope of helium often called an alpha particle). The ^{2} in the $^{2}\mathrm{H}$ indicates that this isotope has 2 nucleons (a nucleon is a particle in the nucleus, either protons or neutrons), in this case 1 proton and 1 neutron. We know it has 1 proton because Hydrogen has 1 proton (that's what makes it Hydrogen). The $^{4}\mathrm{He}$ has 4 nucleons, in this case 2 protons and 2 neutrons (Helium has 2 protons).

As described above, we want to calculate the total mass before the reaction and compare it to the total mass after the reaction. We can do that by using Figure 4.3.

From Figure 4.3, we see that $^{2}\mathrm{H}$ has an average mass per nucleon of about 1.0074 amu. Since $^{2}\mathrm{H}$ has 2 nucleons, we can calculate that it has mass (2 nucleons)(1.0074 amu/nucleon) = 2.0148 amu. We see that $^{4}\mathrm{He}$ has an average mass per nucleon of about 1.001 amu, so with 4 nucleons , it has mass (4 nucleons)(1.001 amu/nucleon) = 4.004 amu. From the reaction

$$^{2}\mathrm{H}+{}^{2}\mathrm{H}\rightarrow{}^{4}\mathrm{He}$$

we see that there is 2.0148 amu + 2.0148 amu = 4.0296 amu before the reaction, and that there is 4.004 amu after the reaction. Since 4.0296 > 4.004, there is more mass before the reaction than after. This means there is more rest energy before the reaction than after. Since energy is conserved, this means that some of the rest energy is converted into kinetic energy. So this reaction "releases" kinetic energy, and is useful for power generation.

With this analysis, we might re-write the reaction as follows:

$$^{2}\mathrm{H}+{}^{2}\mathrm{H}\rightarrow{}^{4}\mathrm{He}+\text{Kinetic Energy}$$

Those of you familiar with chemical reactions might classify this as an exothermic reaction.

On the other hand, elements with large nucleon number A have a larger mass/nucleon than those with intermediate values of A; consequently, kinetic energy can be released by splitting up one of these atoms (fission).

Of the two processes – fission and fusion – fission is a much easier process to achieve in the laboratory or in industrial processes. Large nucleii are naturally unstable, e.g., ^{235}U can spontaneously decay via the process $^{235}U \rightarrow {}^{134}Xe + {}^{100}Sr + {}^{1}n$. Practically, then, the issue boils down to setting things up such that the process can be accelerated when desired, and can be inhibited when unwanted. From that perspective, the concept of a *chain reaction* is relevant. The ideas of nuclear fission and chain reactions – which were pioneered by Lise Meitner, Otto Hahn, Fritz Strassmann and Enrico Fermi in the 1930s – is straightforward: if the neutrons that are released in a fission process bombard another nearby (unstable) nucleus, they can trigger the fission of that nucleus as well. Practically, all that is needed is a large enough density of the unstable ^{235}U and a chain reaction will start. This idea was pursued by the *Manhattan Project* in the 1940s to develop an atomic bomb, the detonation of which was achieved by explosively compressing a Uranium sample to increase its mass above the critical value for a chain reaction. The technique has since been refined with the use of controllable graphite rods (which absorb neutrons) to allow the reaction to proceed in a controlled manner in power generators.

Nuclear fission has a few serious drawbacks: (a) the fuel is expensive (Uranium) and limited in supply. If society were to switch entirely to fission-based power generation, it is estimated that the supply of Uranium would last for only 50-100 years. (b) The by-products of the fission reaction are nucleii which themselves are unstable and radioactive; consequently, the material poses significant environmental and health hazards unless properly stored. (It is possible to extract additional energy from this nuclear "waste", though the environmental and health hazards remain.)

Another drawback of nuclear fission is the concern that reactors could malfunction, such as through a "melt-down" and release massive amounts of radiation (this actually happened to the Chernobyl reactor in the Ukraine in 1986). Other similar operational issues exist. While there had been some confidence that modern engineering designs and safety protocols had reduced the dangers associated with reactor malfunctions, the 2011 Fukushima Daiichi nuclear disaster in Japan is a recent demonstration of the dangers when something goes wrong with this technology. In contrast, nuclear reactors in (non-Soviet) Europe have operated for many decades without significant incident; however in response to Fukushima, there have been severe shifts in European Union nuclear policy. Nuclear power from fission reactions remains a complex and contentious issue, scientifically, technologically, and politically.

Nuclear fusion, by contrast, runs off water (actually, ^{2}H which can be found in water) and produces Helium as a by-product, so waste disposal is less of a problem.[1] The energy production is also much more efficient for this process than for fission, as can be inferred from the steepness of the curve in Fig. 4.3. Estimates are that there is enough ^{2}H (Deuterium) in ocean water to power the world's needs for many thousands of years (if not millions). In fact, nuclear fusion is the power source in stars, including our own Sun. It can be argued that almost all of the Earth's energy sources can be traced back to nuclear fusion.

[1] Some radioactive tritium is released in the process as well, but it is short-lived with a half-life of only 12 minutes; consequently there is no long-term waste problem with the tritium.

Nuclear fusion is not without its problems, though. Specifically, it is very difficult to achieve in a controlled manner. Making a fusion bomb (unfortunately) isn't that difficult (relatively speaking), as a fission explosion can be (and has been) used to compress hydrogen together and cause explosive fusion. But to achieve a controlled fusion reaction is a very difficult procedure that will require a significant amount of ingenuity over the next few decades. This is a *very* important problem – an argument could be made that the future of our modern energy-intensive civilization will depend on developing techniques to achieve cost-efficient fusion power generation. As with anything involving energy and energy policy, it is reasonable to expect that there is no single technological solution to our ongoing energy crisis. In fact, any suite of technological solutions will likely also be insufficient – the energy challenges facing us and those who follow us are too complex to be solved by science alone.

PROBLEMS

1) A nucleus with mass 2.24 GeV/c^2 in its ground state and initially at rest absorbs an incoming photon of unknown energy E_1. After absorbing the photon, the nucleus is raised to an excited state, with mass 5.00 GeV/c^2, and recoils with unknown momentum p_3.
   a) Determine the energy, momentum, and speed of the incoming photon.
   b) Determine the energy, momentum, and speed of the excited nucleus after it absorbs the incoming photon.
   c) Note that this absorption of a photon by a nucleus is the exact opposite of the emission of the photon by a nucleus done as Example 1 in this chapter (the ground state nucleus has mass 2.24 GeV/c^2 and the excited nucleus has mass 5.00 GeV/c^2 in both cases.) Compare the speed for the recoiling nucleus in Example 1 with the speed of the nucleus you just calculated. Are the recoil speeds the same for emission and absorption?

2) A particle of mass m_1 = 9 GeV/c^2 and energy E_1 = 15 GeV approaches a stationary particle of mass m_2 = 5 GeV/c^2. The particles collide and form a single particle of mass m_3. Determine E_3, p_3, and m_3.

3) An incident proton, mass m = 938.27 MeV/c^2, strikes a target proton at rest with just enough energy to create an ***electron-positron*** pair. (The two protons are still present after the collision.) A positron is the antiparticle of an electron; both the electron and positron have masses 0.511 MeV/c^2. Calculate the minimum energy needed by the incident proton in the frame where the target proton is initially at rest. (Hint: After the collision, both protons and the electron-positron pair all move together with the same velocity.)

4) A particle of mass 3.0 MeV/c^2 and momentum 1.0 MeV/c hits and sticks to a particle of mass 2.0 MeV/c^2, initially at rest.
   a) Find the mass of the composite particle and its velocity.
   b) How much kinetic energy is converted to mass?

5) A photon of momentum 2.0 MeV/c traveling along the positive x-axis strikes a stationary particle of mass 4.0 MeV/c^2. After the collision, there are simply two photons: photon γ_1 travels backward, along the negative x-axis and photon γ_2 travels forward, along the positive x-axis. Find the energies of γ_1 and γ_2 after the collision.

6) A particle of mass 1.00 MeV/c^2 is moving to the left when it absorbs a photon of momentum 7.5 MeV/c moving to the right. The absorption stops the particle and excites it to a new total energy E_f. Determine E_f and m_f.

7) Particle A of mass 7.5 GeV/c^2 and energy 12.5 GeV moving to the right collides with stationary particle B of mass 6.0 GeV/c^2. The result is a stationary particle C of unknown mass m_C and a photon γ of unknown energy E_γ. How do you know the photon is moving to the right? Determine m_C and E_γ.

8) A 3.0 GeV photon hits a stationary particle A (mass m_A). After the collision there are simply two particles. Particle B has mass 1.0 GeV/c^2 and unknown momentum p_B. Particle C has mass 3.0 GeV/c^2 and momentum 4.0 GeV/c and moves in the same direction as the original photon. Determine m_A and p_B.

9) A particle of mass 6.0 GeV/c^2 and momentum 8 GeV/c is moving to the right and collides with a stationary particle, also of mass 6.0 GeV/c^2. The two particles annihilate each other, resulting in two photons: one moving the right and one moving to the left. Determine the energy of the left-moving photon and the energy of the right-moving photon.

10) Particle A of unknown mass m_A moves to the right with speed 0.6c. At some instant, it decays into two particles: stationary particle B of unknown mass m_B and a 3 MeV photon moving to the right. Determine m_A and m_B.

11) In a nuclear reaction, two deuterium nuclei each of mass 1875.61 MeV/c^2 combine to form a single helium nucleus of mass 3727.38 MeV/c^2. Is rest energy converted to kinetic energy or vice-versa? Support your answer with a calculation. Would you classify this as a fusion reaction or a fission reaction?

12) In a nuclear reaction, a slow neutron causes a uranium nucleus (mass = 218,943.42 MeV/c^2) to split into a Barium nucleus (mass = 131,261.73 MeV/c^2) and a Krypton nucleus (mass = 85,629.32 MeV/c^2), plus <u>two</u> excess neutrons (actually 3 including the original neutron, but that is present before the process as well), each of mass 939.57 MeV/c^2. Calculate the energy converted from mass to kinetic energy in this process. Would you classify this as a fusion reaction or a fission reaction?

13) Based on the plot in Figure 4.3, explain why a fusion reaction is a more efficient power source ("pound for pound" or by mass) than a fission reaction.

14) Nitrogen (A = 14) comprises more than 75% of the atmosphere and is readily available. Would Nitrogen make a good fuel source for a fission reactor to generate energy? Use Figure 4.3 to explain why this suggested fission process will or will not work to produce energy.

15) After a supermassive star has run out of Hydrogen to fuse, it starts fusing Helium into heavier elements, then fusing those into heavier elements, etc., until it gets to iron (Fe). Up until this point, the fusion reactions produce kinetic energy in the form of rapidly moving atoms and molecules and photons. The star is effectively exploding, but its massive gravitational field keeps the explosion contained, and this balance results in a stable star. But after the star has fused its materials into iron, it stops producing kinetic energy and the gravitational interaction causes it to contract very suddenly; the massive decrease in gravitational energy results in an uncontained explosion. This is one way in which a star can go supernova. Use Figure 4.3 to explain what is so special about iron. Why can't the star produce additional kinetic energy via fusion of iron with itself or lighter elements?