Math Lab 10: Differential Equations and Direction Fields

Complete before class Wed. Feb. 28; Due noon Thu. Mar. 1 in class

Goals:

1. Gain exposure to terminology and notation associated with differential equations.
2. Learn to sketch direction fields and draw solution curves for particular differential equations by hand and by Desmos.
3. Verify proposed solutions to particular differential equations.

Instructions: Same as previous labs; see Handout folder in program file share.

INTRODUCTION

> (from Stewart p. 493) "Perhaps the most important of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information."

In this Math Lab (our last of the quarter, and of the year!), you will gain some further experience with differential equations in advance of our formal study of them in Chapter 7 during weeks 18 and 19.

You have already encountered differential equations, frequently in physics, particularly in kinematics and dynamics but also in circuits (e.g. the charging/discharging RC circuit), and will encounter them shortly in chemistry in the context of kinetics. Differential equations also synthesize the two main parts of single-variable calculus: the relevant rate(s) of change in the mathematical model involve derivatives, and when there is an analytical solution, it often involves integration. You've practiced solving differential equations analytically (though perhaps those encounters weren't explicitly stated to be solving differential equations) and numerically (using the Euler-Aspel-Cromer method, in Physics Lab 9).

PART 1: BRIEF PRIMER ON DIFFERENTIAL EQUATIONS

Perhaps not surprisingly, a differential equation is an equation that involves a derivative of an unknown function. Solving the differential equation means finding that unknown function (or family of functions).

For example, say that we know that $y' = dy/dx = f(x,y)$ for some known function $f(x,y)$, but the function $y = F(x,y)$ is unknown. Then solving the differential equation $y' = f(x,y)$ means finding that unknown function $y = F(x,y)$.

1. Below is one of the simplest differential equations I can think of. Given this differential equation, what does your prior knowledge tell you that $y =$? Write down your best proposal for y, take its derivative, and show that $y' = 1$ for your proposed y.

$$y' = \frac{dy}{dx} = 1$$

$y =$

2. Did you propose $y = x + C$ where C is some constant? If so, great. If you forgot the C, go back and fix it now, and write yourself a note in the box that will remind you that in general, a solution to a differential equation is a family of functions. You might recognize that in this case, solving the differential equation involved finding the anti-derivative as you have done many times before.

3. Repeat for the following, to determine a family of functions y whose y' is as given:

$y' = \frac{dy}{dx} = 0$	$y' = \frac{dy}{dx} = x$	$y' = \frac{dy}{dx} = y$ (note that this is asking for a function whose derivative is itself)

4. Hopefully you got $C, \frac{1}{2}x^2 + C$, and Ce^x. For the last one, you might have thought it was $e^x + C$, but note that in this case $y' = e^x \neq e^x + C$. If you're not sure about these, check with your neighbor or with TA/faculty.

5. You saw that there was a solution to the differential equation $y' = x$ and $y' = y$. What about for $y' = x + y$? As a first guess, I would try $y = \frac{1}{2}x^2 + e^x$ and see what happens. Try for yourself and confirm that this doesn't work.

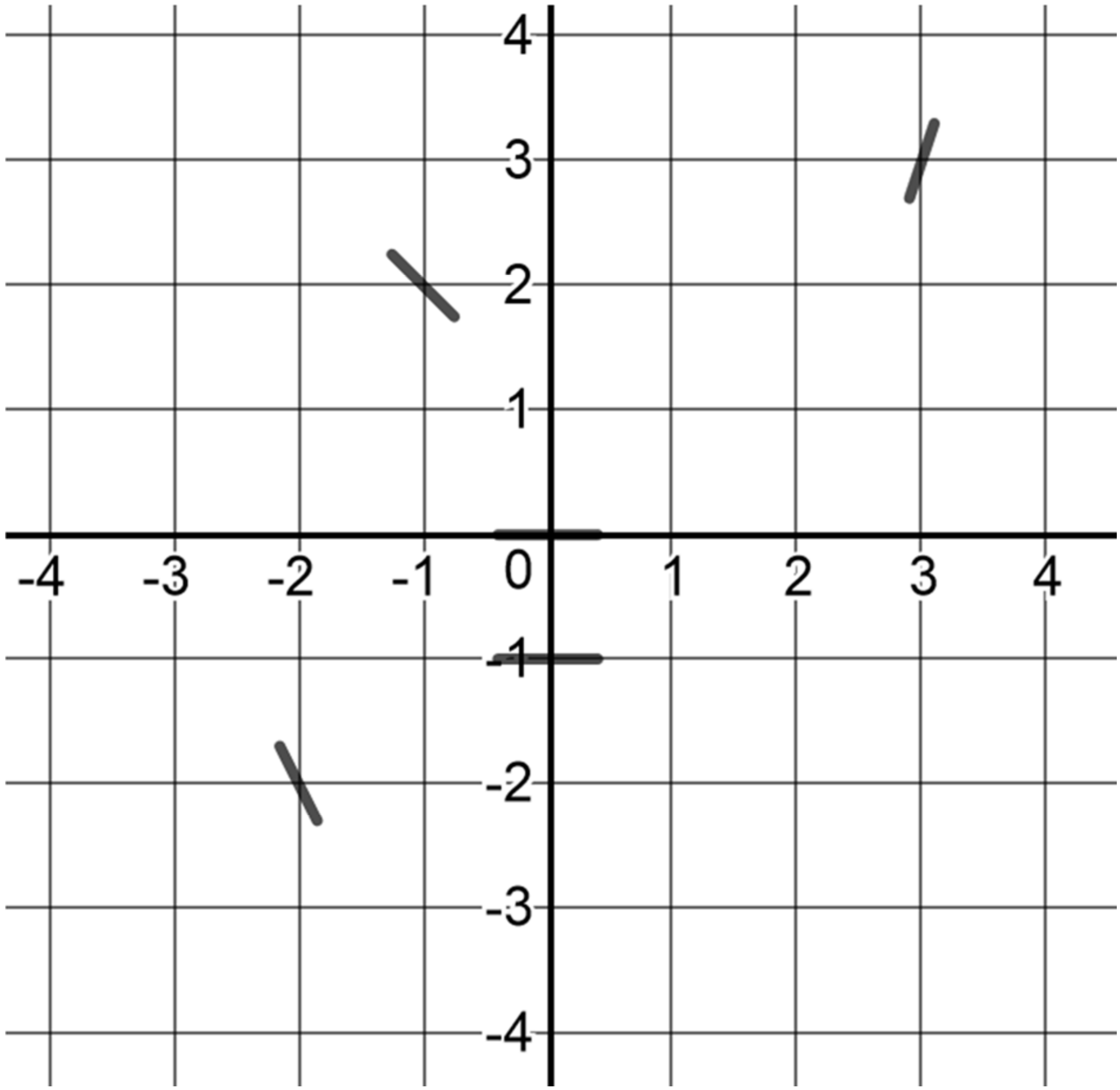
PART 2: DIRECTION FIELDS, BY HAND

It turns out that there is an analytic method to find the solution to the differential equation $y' = x + y$, but we won't worry about that for now (an approach is suggested on the board, should you be interested in trying it after you have completed the lab). Instead, we will explore a graphical method that helps us visualize the shapes of the solutions to a differential equation, a graphical method call direction fields or slope fields. You will learn how to draw direction fields by hand. That method is a bit tedious, but I hope will give you a foundation for understanding what you will see when we use Desmos to draw direction fields for us.

6. Recall that the (first) derivative y' gives us the slope of y at any point. Consider again $y' = x$. For the point $(0, 0)$, where $x = 0$ and $y = 0$, then $y' = x = 0$. For the point $(0, -1)$, where $x = 0$ and $y = -1$, then $y' = x = 0$ (again). For $(-1, 2)$, then $y' = x = -1$, etc. In this simple case, the solution does not depend on the y values. Fill out the table below, which has some of the entries already filled in (hopefully you will see an interesting pattern that you can use to save some time):

	$x = -3$	$x = -2$	$x = -1$	$x = 0$	$x = +1$	$x = +2$	$x = +3$
$y = +3$							
$y = +2$			-1				
$y = +1$							
$y = 0$				0			
$y = -1$				0			
$y = -2$							
$y = -3$							

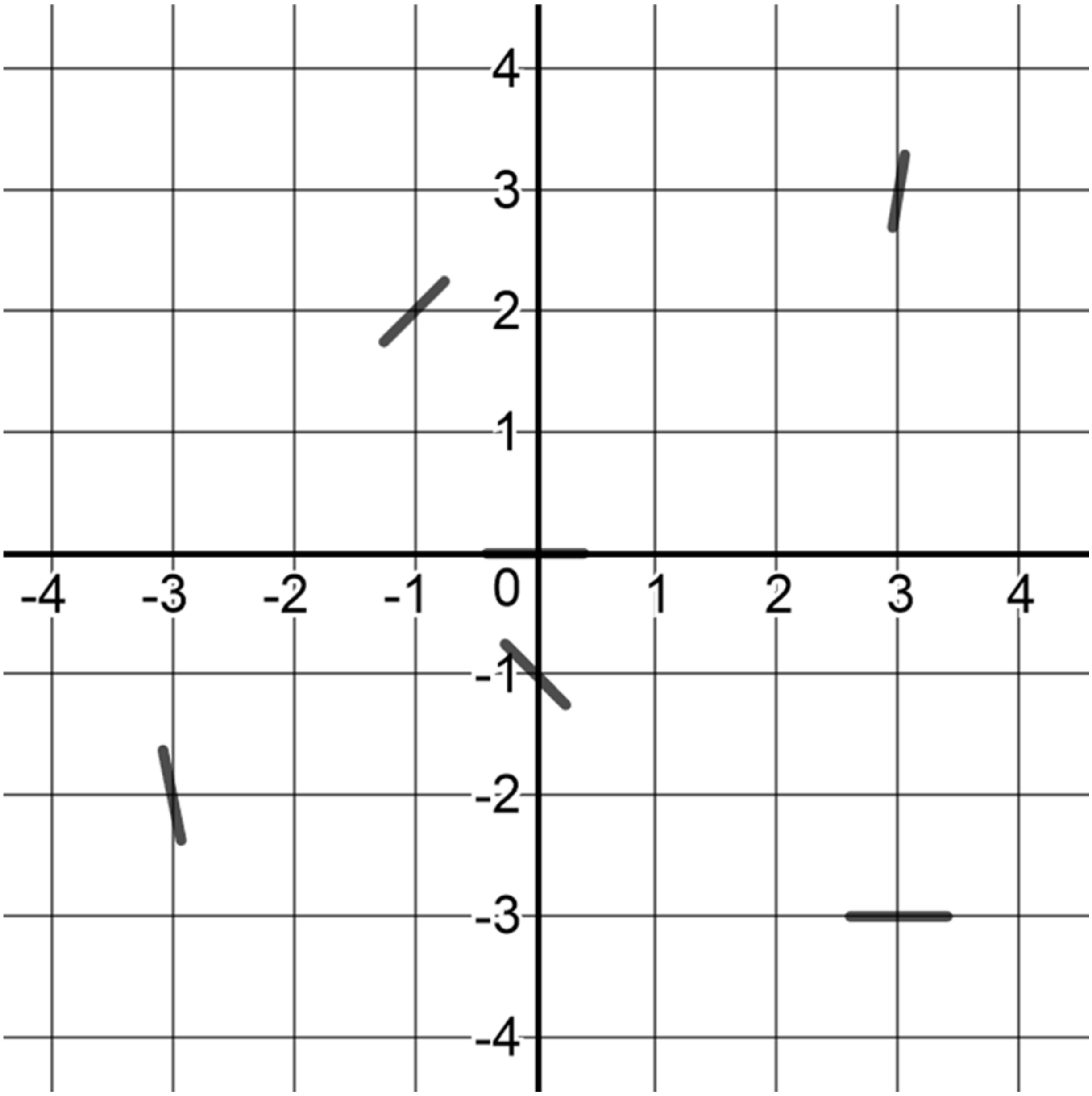
7. On the graph on the next page, you will plot each slope for each point <u>in pencil</u>. Use a straight edge to draw straight line segments. Draw the slope as accurately as you can, using the grid lines to help. Some are already plotted for you as a model. Note that we are just drawing short line segments to represent the slopes, but those slopes are accurate.



8. Repeat, but this time for $y' = x + y$. Again, a few entries are filled out for you (and again, hopefully you will see an interesting pattern that you can use to save time).

	$x = -3$	$x = -2$	$x = -1$	$x = 0$	$x = +1$	$x = +2$	$x = +3$
$y = +3$							$+6$
$y = +2$			$+1$				
$y = +1$							
$y = 0$				0			
$y = -1$				-1			
$y = -2$	-5						
$y = -3$							0

9. Plot the direction field below, again **using pencil**.



10. If you haven't already, compare your direction fields to your neighbors. As needed, make any corrections/updates (see why I asked you to use pencil?).

11. https://www.youtube.com/watch?v=F7wKjCCqehA is a reasonably short video tutorial I found on direction fields that seems acceptable; watch it now and jot down any useful notes in the space below.

PART 3: DIRECTION FIELDS, BY DESMOS

It was likely tedious to draw even just a few points of the direction field by hand. Luckily, we can use a computer to plot direction fields for us. We'll use Desmos. You might want to keep the plots you produce in this section; recall that you can do that either by screenshot and then pasting into Word or PowerPoint (or equivalent), or you click on the Share Graph icon (next to the Help icon) and click on export image, which will give you several options for saving the graph and then you can insert it as a picture into Word, PowerPoint, etc.

12. Go to the following Desmos calculator **https://www.desmos.com/calculator/422bzlk5yr**
In input line 2, you should see $f(x, y) = x$, so this should be the direction field (slope field) corresponding to what you drew by hand for step 7 (though for a larger plotting window), and what you saw in the video. Confirm this.

13. Change the entry in input line 2 to be $f(x, y) = x + y$ and confirm that this is matches what you do in step 9 and saw in the video.

14. Return to $y' = f(x, y) = x$. Back in part 1, you showed that the solution to this was $y = \frac{1}{2}x^2 + C$. Enter this into (what should be the empty) input line 3, and turn on the C slider. Adjust the C slider. What do you notice about the solution curves and the direction field?

15. Hopefully you noticed that the direction field suggests the solution curve. Hopefully this makes sense: at any point, the direction field shows the tangent to the function which is one of the solutions to the differential equation. If you draw enough tangent lines, close enough together, you should get a sense of the "original" functions that are solutions to the differential equation.

16. Change so that $f(x, y) = 0$ and the solution curve to be $y = C$ (you found this in step 3). Adjust the C slider. Any surprises?

17. Change so that $f(x, y) = 1$ and the solution curve to be $y = x + C$ (you found this in step 1). Adjust the C slider. Any surprises?

18. Change so that $f(x, y) = y$ and the solution curve to be $y = Ce^x$ (you also found this in step 3). Adjust the C slider. Any surprises?

19. Change so that $g(x, y) = -y$ (which you did not do previously, though you may be familiar with it). Look at the direction field, and in particular, compare it to the direction field for $g(x, y) = y$. Can you guess what the solution curve will be?

20. Did you guess $y = Ce^{-x}$? Good. Test it out and see. Any surprises?

21. Return to $g(x, y) = x + y$. Hide the input in line 3, as we don't (yet) have a formula for the solution curves. It turns out that $y = Ce^x - x - 1$ is the solution to $y' = x + y$. By taking the derivative of $y = Ce^x - x - 1$, verify that it is a solution to $y' = x + y$.

22. In input line 3, enter $y = Ce^x - x - 1$, and adjust the C slider. Any surprises?

PART 4: INTEGRATING AND EXTENDING

23. For now, don't use Desmos. Consider the differential equation $y' = -x/y$. Are you able to quickly write down its solution, like you could for $y' = 0$, $y' = 1$, $y' = x$, $y' = y$, or even $y' = -y$? If so, do so in the space below, but if not, that's ok; just write "Not yet!".

24. Can you picture the direction field in your head? It's ok if not but cool if you can. Enter $g(x, y) = -x/y$ into Desmos. What do you notice about the direction field? What does that suggest about solution curves?

25. Enter $x^2 + y^2 = r^2$ into input line 3, turn on the r slider and adjust it. Any surprises?

26. By taking the derivative of $x^2 + y^2 = r^2$, show that these are solutions to $y' = dy/dx = -x/y$. Hints: solve for y^2, then take the derivative using the chain rule (i.e. use implicit differentiation).

27. Starting from $dy/dx = -x/y$, show that $y\,dy = -x\,dx$. Then, integrate both sides and see what you get.