Math Lab 9: **Volumes$^{\text{Volumes}^{\text{Volumes}}}$**, or I ♡ Volumes

Due noon Thu. Feb. 15 via the Workspace: Math Lab 9 folder

Goals: (will vary depending on your choice)

1. Use calculus to find volumes of cones, spheres, and other solids of revolution.
2. Explore classical methods for finding volumes of cones and spheres without integration.
3. Explore spheres in higher dimensions.

Instructions: Same as previous labs; see Handout folder in program file share.

INTRODUCTION

In this lab, you may work by yourself or in small groups. In either case, work at whiteboards, either at your table or standing up. Take pictures of your work in landscape mode (if you/your group doesn't have a camera, I can take pictures for you); make sure your initials are clearly visible in every picture you take. After lab, insert your pictures into a PowerPoint file (one picture per slide that fills the slide for easy visibility, this is why landscape mode) and save the file to the appropriate place in the Workspace: Math Lab 9 folder; name the file using the following example: Part 3 Emmy Maria Sophie (i.e. start with the Part and then list first name(s) in alphabetical order). Except for assembling your pictures into your PowerPoint files, you are not intended to work on this outside of class time; instead use the time you might have spent out of class to work on your Energy Project Literature Review. It's ok if what you save and share is incomplete.

In this lab, you will investigate volumes in a variety of contexts. All students should complete part 1, which reviews how to use calculus to find the volume of two particular solids of revolution: a cone and a sphere, using both the disk/washer and shell methods. Then, you will choose from a variety of different activities, completing at least one of them, and more if you have time or interest. After completing Part 1, quickly skim over Part 2: Napkin Rings, Part 3: Complementary Coffee Cups, Part 4: Classical Methods, and Part 4: Hyperspheres, and choose the one you find most interesting to work on (they are listed here according to my tastes and also space considerations, with Part 2 and Part 3 being perhaps the most straightforward but also perhaps somewhat less interesting, and with Part 3 and Part 4 more involved and perhaps somewhat more interesting). As your time and interest permits, work on other parts.

PART 1: CALCULUS METHODS TO FIND THE VOLUME OF A CONE AND A SPHERE

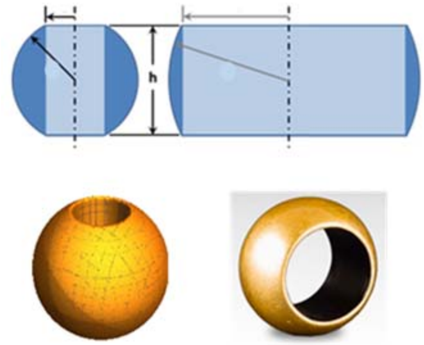
<u>0. SPEND NO MORE THAN 30 MINUTES ON THIS PART. STOP WORKING ON IT AT 12:15.</u>

1. In class, we went over using the disk/washer method and the shell method to find the formula for the volume of a cone with base radius r and height h. Working by yourself or in small groups, reproduce that work (if possible without referring to your notes). Work at whiteboards either at your table or standing up. Take pictures of your work following the guidelines above.

2. If time permitted, we might also have gone over how to find the volume of a sphere of radius r using both disk and shell methods. In any case, working by yourself or in small groups, find the formula for the volume of sphere of radius r using the disk and shell methods (if possible without referring to notes or book). Work at whiteboards either at your table or standing up. Take pictures of your work following the guidelines above.

After completing Part 1 (or having run out of time),
quickly skim over the remaining Parts and
choose one to work on for the remainder of the lab time.

PART 2: NAPKIN RINGS

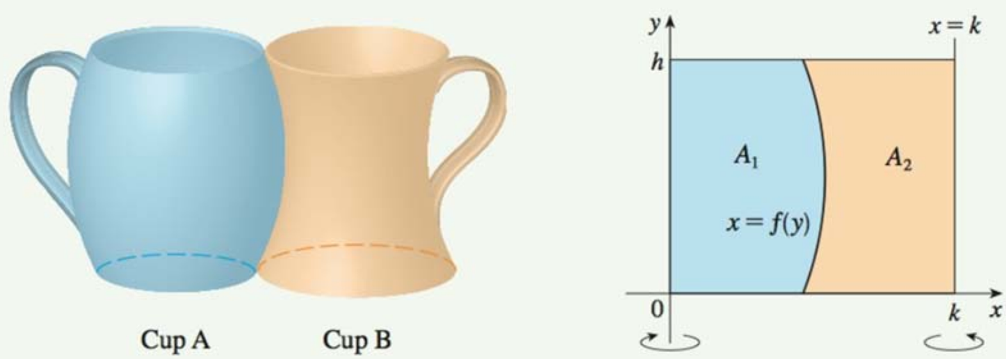


Consider 2 spheres; the smaller with radius a and the larger with radius b. In each sphere, a cylindrical hole of height h is drilled out. Notice that the holes have the same height but since the spheres have different radii, the radii of the holes are also different, as shown in the top 2 pictures in the figure on the right. So each sphere has a cylindrical hole and the end "caps" removed. This leaves behind a "napkin ring" shape, some examples of which are illustrated in the bottom two figures. Which napkin ring has the larger volume: the one from the sphere of radius a, or the one from the sphere with radius b?

Take pictures, etc. as described above.

PART 3: COMPLEMENTARY COFFEE CUPS[1] (Stewart p. 475)

Suppose you have a choice of two coffee cups of the type shown, one that bends outward and one inward, and you notice that they have the same height and their shapes fit together snugly. You wonder which cup holds more coffee. Of course you could fill one cup with water and pour it into the other one but, being a calculus student, you decide on a more mathematical approach. Ignoring the handles, you observe that both cups are surfaces of revolution, so you can think of the coffee as a volume of revolution.



1. Suppose the cups have height h, cup A is formed by rotating the curve $x = f(y)$ about the y-axis, and cup B is formed by rotating the same curve about the line $x = k$. Find the value of k such that the two cups hold the same amount of coffee.

2. What does your result from Problem 1 say about the areas A_1 and A_2 shown in the figure?

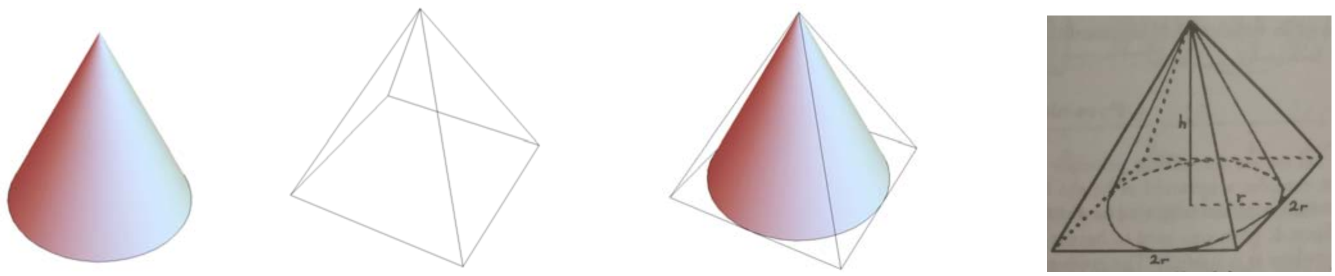
Take pictures, etc. as described above.

[1] I feel like in a Pacific Northwest calculus class, one is obliged to ask a coffee question if presented with the opportunity. No obligation to answer the question, of course. You pick.

PART 4: CLASSICAL METHODS[2]

We've used integration to find formulas for the volumes of a cone and a sphere. But these formulas were known to many cultures well before the modern development of integral calculus. In this part of the lab, you will investigate how to deduce these formulas without formal integration. You will use a related calculus idea: that a volume can be considered to be made up of slices. Take pictures, etc. as described above.

1. Consider a cone with radius r (and thus diameter $2r$) and height h. Also, consider a square pyramid, with square base length $2r$, also of height h. Both are shown separately, and also with the cone inside the pyramid (from two perspectives).

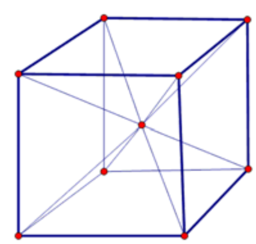


2. Consider the base of the cone and the pyramid. What is the area of the pyramid base? What is the area of the cone base? Show that the ratio of those areas is $\pi/4$.

3. Choose any cross sectional slice that is parallel to the base. Convince yourself (and each other) that the ratio of the areas of the cone cross-section to the square pyramid cross-section is again $\pi/4$.

4. Now, using this the fact that the ratio of the areas of cone cross-section to square-pyramid cross section (for cross-sections parallel to the base) is equal to $\pi/4$ and the idea that the volume is made up of slices to convince yourself (and each other) that this means the ratio of their volumes is also $\pi/4$. Therefore $V_{cone} = \frac{\pi}{4} V_{pyramid}$.

5. This might be an interesting result, but it doesn't really help us unless we know a formula for $V_{pyramid}$. However (as you may have suspected), we can figure this out, also without using integration. Hint: start by constructing a cube with side length a. We know the volume of this cube, by definition, is $V_{cube} = a^3$. Draw the body diagonals that connect opposite corners. Can you see how to proceed? Try it out. If you get stuck, read the next part.



6. From the construction, you can hopefully see that you have split the cube up into 6 pyramids (with square bases), each of which has square base length a but height $\frac{a}{2}$. Since the total volume is a^3, then each pyramid must have volume $\frac{1}{6}a^3$. A little rearranging gives $V_{pyramid} = \frac{1}{6}a^3 = \frac{1}{3}\frac{1}{2}a\,a\,a = \frac{1}{3}\left(\frac{1}{2}a\right)a^2$. Recognizing that a^2 is the area of the base and $\frac{1}{2}a$ is the height, we see that the volume of the pyramid in this particular case is $\frac{1}{3}$(base area)(height).

7. What about when the height is not equal to half the base length? If you take the cube and stretch it (say vertically) by a stretch factor s where $s = 2h/a$, so that the new height is $2h$, then you've changed the shape from a cube to a rectangular solid. Stretching the cube in this way increases the volume exactly by s, such that $V_{rectangular} = sV_{cube} = \frac{2h}{a}\,a^3 = 2ha^2 = a^2(2h)$; as expected, the volume of this rectangular solid is the base area times the height. This stretch factor s also stretches the *height* of the top and bottom pyramids, changing $\frac{1}{2}a$ to h. It also changes one of the base sides of the other four side pyramids, from a to $2h$. This means that all six pyramids still have the same volume as each other, so now we have $6V_{pyramid} = V_{rectangular} = a^2(2h) \rightarrow V_{pyramid} = \frac{1}{6}\,a^2(2h) = \frac{1}{3}\,a^2h$. We again see that the volume of the pyramid is $\frac{1}{3}$(base area)(height).

8. Now, returning to our earlier result (from step 4) that $V_{cone} = \frac{\pi}{4}V_{pyramid}$, (where $V_{pyramid} = \frac{1}{3}\,a^2h$ from step 7) and

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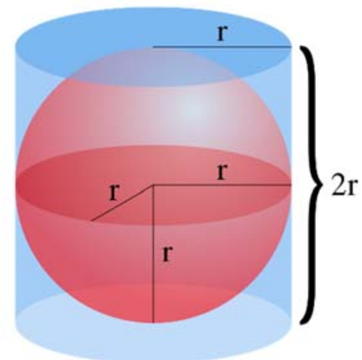
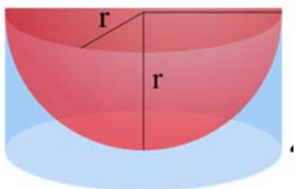
[2] This part is based on Lab 10: Volumes and Hypervolumes in *Applications of Calculus*, ed. Philip Straffin, 1993, MAA; some figures taken from that source.

recognizing that in this case, $a = 2r$, find V_{cone}.

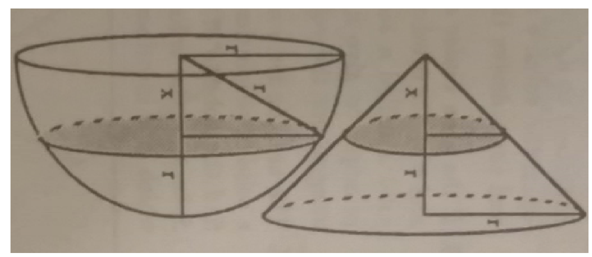
That's how you find the formula for the volume of a cone! Take that, integration! Also, thank you integration for giving us another way to find it!



9. Now, on to the sphere. Consider the figure shown to the right[3], which shows a sphere of radius r inscribed in a cylinder, also of radius r and height $2r$. Consider just the bottom half of the sphere and cylinder (now of height r), shown below. Also consider a cone of height r and base radius r, also inscribed in the cylinder (but not shown in the figure to the left).



10. The figure to the right shows the hemisphere and the cone, side-by-side for clarity, but you might find it helpful to consider them inscribed in the same cylinder as shown above. (My apologies that the labels are sideways; if it helps, turn the paper by 90 degrees counter clockwise.)



Consider the very bottom of the cylinder (and thus the bottom of the hemisphere and the cone). What is the cross-sectional area of the cylinder? What is the cross-sectional area of the cone? What is the cross-sectional area of the hemisphere (note that the hemisphere just touches at a single point (its 'pole') at the bottom of the cylinder). Similarly, consider the very top of the cylinder (and thus the top of the hemisphere and the tip of the cone). What is the cross-sectional area of the cylinder? What is the cross-sectional area of the hemisphere? What is the cross-sectional area of the cone (note that the cone has come to its point at the top)?

11. Did you notice that the sum of the cross-sectional areas of the hemisphere and cone at the bottom equaled the cross-sectional area of the cylinder? Did you notice that the sum of the cross-sectional areas of the hemisphere and cone at the top equaled the cross-sectional area of the cylinder?

12. Look at the figure again, and consider the cross-sectional areas shown, at some height x from the top of the cylinder. Calculate the cross-sectional area of the hemisphere in terms of r and x (hint: use Pythagorean theorem). Calculate the cross-sectional area of the cone in terms of x (hint: use trigonometry/similar triangles). Add these together and simplify.

13. Did you find that the sum of these cross-sectional areas equaled the cross-sectional area of the cylinder? This is consistent with what you found in step 11.

14. Note that you did this for some generic x. This means that you have shown that for any cross-sectional slice, the sum of the area of the hemisphere and the cone is equal to the area of the cylinder. Thus, by the principle of slicing, you've shown that the sum of the volume of the cone plus the hemisphere equals the volume of the cylinder of radius r and height r. Use this, along with what you know about the volume of cylinders and cones, to find the formula for the volume of a sphere of radius r.

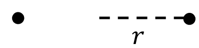
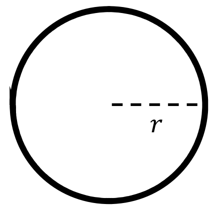
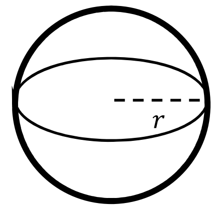
15. Yay!

[3] By Original: AndertxumanVector version: CheCheDaWaff - This file was derived from Esfera Arquímedes.jpg:, Public Domain, https://commons.wikimedia.org/w/index.php?curid=55042681

PART 5: Volume and Hypervolume[4]

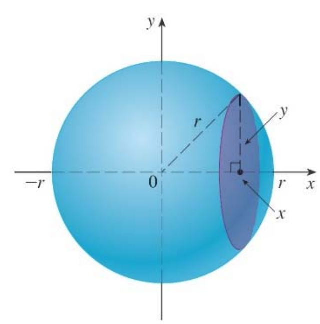
In this part, you will try to generalize the integration methods we have used for finding the area of a circle and the volume of a sphere to try to find the "hypervolume" of a "hypersphere". Take pictures, etc. as described above.

Consider the following table, which summarizes some results we have encountered previously:

"name"	"diameter" 1-D ball	circle 2-D ball	Sphere 3-D ball	"hypersphere" 4-D ball
dimension	1	2	3	4
representation				?
equation of surface	$x^2 = r^2$	$x^2 + y^2 = r^2$	$x^2 + y^2 + z^2 = r^2$	$x^2 + y^2 + z^2 + w^2 = r^2$
"volume"	$V_1(r) = u_1 r = 2r$	$V_2(r) = u_2 r^2 = \pi r^2$	$V_3(r) = u_3 r^3 = \frac{4}{3}\pi r^3$	$V_4(r) = u_4 r^4$

1. Let's review two calculations and see if a pattern emerges that we can use:

For a circle, $x^2 + y^2 = r^2 \rightarrow y = \sqrt{r^2 - x^2}$ (this give us the upper half of the circle of radius r). To find its "volume" (in this case, we call it area) by Riemann rectangles, which we can imagine as slices of the circle: $A = V_2 = \int_{-r}^{r} 2\sqrt{r^2 - x^2}dx$. Draw a picture to support this. The 2 inside the integrand comes from the fact that our function $y = \sqrt{r^2 - x^2}$ is only the upper half of the circle so we need to double to account for that, taking advantage of symmetry. We can also see from our picture that we have an even function, so we can take more advantage of symmetry: $V_2 = \int_{-r}^{r} 2\sqrt{r^2 - x^2}dx = 2\int_0^r 2\sqrt{r^2 - x^2}dx = 4\int_0^r \sqrt{r^2 - x^2}dx$. The standard approach for integrals involving $\sqrt{r^2 - x^2}$ is to try a trigonometric substitution $x = r\sin\theta$, so $dx = r\cos\theta \; d\theta$. This transforms $\int_0^r \sqrt{r^2 - x^2}dx$ into $\int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2\theta} \; r\cos\theta \; d\theta = r^2\int_0^{\pi/2} \cos^2\theta \; d\theta$. This can be evaluated using a trig identity or by parts. If you'd like to practice this, you are welcome to fill in the steps, but once the integral is set up, it can be evaluated in Mathematica or any other equivalent. Evaluating this integral gives us $\pi/4$, which combined with the 4 from the symmetry arguments and the r^2 from the trig substitution gives us $V_2 = \int_{-r}^{r} 2\sqrt{r^2 - x^2}dx = 2\int_0^r 2\sqrt{r^2 - x^2}dx = 4\int_0^r \sqrt{r^2 - x^2}dx = 4r^2\int_0^{\pi/2} \cos^2\theta \; d\theta = \pi r^2$, as expected.



For a sphere: $x^2 + y^2 + z^2 = r^2 \rightarrow y^2 + z^2 = r^2 - x^2 \rightarrow \sqrt{y^2 + z^2} = \sqrt{r^2 - x^2}$. Looking at Figure 4 in section 6.2, this tells us that at any location x within this sphere, the cross-sectional slice of the sphere is a circle in the yz-plane, with radius $\sqrt{y^2 + z^2} = \sqrt{r^2 - x^2}$.

Following what you did in Part 1 or as laid out in Exampe 1 in section 6.2, we find its volume by slicing the sphere and adding up the slices: $V_3 = \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \pi\,(r^2 - x^2)dx$. Here, the π and the $\left(\sqrt{r^2 - x^2}\right)^2$ comes from using the disk method: we are building up our 3D-ball by using slices that are circles. Then, taking advantage of symmetry again, we get $V_3 = \int_{-r}^{r} \pi\,(r^2 - x^2)dx = 2\pi\int_0^r (r^2 - x^2)dx$. This is straightforward to integrate (and you've already done this in Part 1) to arrive at $V_3 = \frac{4}{3}\pi r^3$.

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[4] This part is based on Lab 10: Volumes and Hypervolumes in *Applications of Calculus*, ed. Philip Straffin, 1993, MAA; some figures taken from that source.

2. Let's review our review, and highlight some pieces.

For a 2D-ball (circle): $V_2 = \int_{-r}^{r} 2r\, dx$, where we used $r \to \sqrt{r^2 - x^2}$.

For a 3D-ball (sphere): $V_3 = \int_{-r}^{r} \pi r^2\, dx$, where we again used $r \to \sqrt{r^2 - x^2}$.

If you look at the summary table at the beginning of this section again, note that $V_1(r) = 2r$ and $V_2(r) = \pi r^2$.

So, that means that

$V_2 = \int_{-r}^{r} V_1(\sqrt{r^2 - x^2})\, dx = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx$ and

$V_3 = \int_{-r}^{r} V_2(\sqrt{r^2 - x^2})\, dx = \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2\, dx.$

We built our circle out of rectangles. We built our sphere out of circles. We propose that we can build our "hypersphere" (our 4D-ball) out of spheres (3D-balls), following the same pattern:

$$V_4 = \int_{-r}^{r} V_3(\sqrt{r^2 - x^2})\, dx = \int_{-r}^{r} \frac{4}{3}\pi\left(\sqrt{r^2 - x^2}\right)^3\, dx$$

Using symmetry, pulling out constants, and condensing, this becomes:

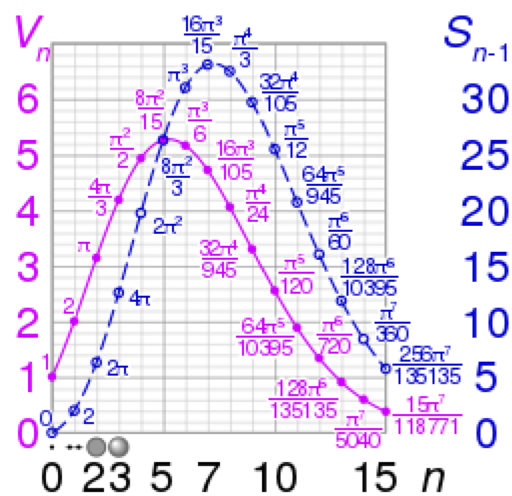
$$V_4 = \frac{8}{3}\pi \int_{0}^{r} (r^2 - x^2)^{3/2}\, dx$$

This integral is evaluable; you are welcome to try it by hand using the same techniques you would use for the circle, though I recommend seeing if you can get your computer to find it for you (you might want to do the trigonometric substitution if your computer can't do it as written above; it's more likely that your computer can evaluate the rest of the way if you do some transformation at the beginning). What do you find for V_4?

3. Hopefully you found $V_4 = \frac{\pi^2}{2} r^4$. What a gorgeous result!

4. Now, use V_4 to find V_5. (You can check your result below.)

5. You could next find V_6 using V_5, V_7 using V_6, etc. At the Wikipedia entry for "Volume of an n-ball", there's a nice graph of V_n vs. n for n-balls of radius $r = 1$. From this graph, we can see that $V_5 = \frac{8\pi^2}{15} r^5$ (you read the coefficient off the graph, and we know that a 5D-ball would go as r^5). Notice a remarkable feature: as the dimensionality n increases, the volume of the n-ball of radius 1 increases up till $n = 5$, at which point it starts to decrease!



Why is that? Does this tell us something about the universe we live in, or at least the universe we can conceptualize?

Another interesting feature is suggested by the choice to connect the data points with a curve. Does an n-ball with $n = 3.5$, for example, exist? So a 3.5D-ball?

6. Up till now, we have mostly encountered and used mathematics as a tool to answer questions from other disciplines. Here, we see mathematics generating its own questions. Hope you enjoyed it!