Math Lab 8: Electric Fields – Integrating Continuous Charge Distributions II

Due noon Thu. Feb. 1 in class

Goals:

1. Learn to use *Mathematica* to plot functions and to numerically evaluate definite integrals, by exploring functions that do not have elementary anti-derivatives.
2. Practice setting up integrals to calculate the electric field due to continuous charge distributions.

Instructions: Same as previous labs; see Handout folder in program file share.

INTRODUCTION

This lab continues and extends last week's Math Lab 7, this time for integrals which must be evaluated numerically, for cases where no elementary anti-derivatives exist. This lab begins with checking on the final results from last week's lab. The lab continues with a brief exploration of functions which don't have elementary anti-derivatives as a way to learn how to use Mathematica's numerical integration capabilities. Then, you will follow along with the procedures you practiced last week to set up integrals to calculate the electric field for a continuous ring of charge anywhere in the plane of the ring (in class and in your text, you've seen the special case of the electric field on the axis of the ring, which can be found analytically). Finally, you'll be challenged to investigate how to handle non-uniform charge distributions.

PART 0: CHECKING ON FINAL RESULTS FROM MATH LAB 7

1. At the end of Math Lab 7, you considered a 0.30 m long rod with 6 nC of charge uniformly distributed over its length that lay along the x-axis, with its right end at the origin. You were to determine E_x and E_y at the field point (0 m, 4 m) by following the standard strategy to find the definite integrals and modifying the provided MMA notebook to evaluate the definite integrals. Many of you struggled with this.

2. If (using MMA) you found $E_x = 0.126031$ and $E_y = 3.36555$ (both in N/C, and both reported with too many digits), great! Please move on to Part 1. Otherwise, please continue with this part.

3. What definite integrals did you find? Hopefully you have these in your notes or you saved your modified MMA notebook and can get it from that. If not, hopefully you will consider saving this kind of work in the future, and you luckily get to practice these skills again. Fill in the blanks.

$$E_x = k\int_{\square}^{\square} \_\_\_\_\_\_\_\_\_\_\_\_ \, d\_\_\_ \qquad\qquad E_y = k\int_{\square}^{\square} \_\_\_\_\_\_\_\_\_\_\_\_ \, d\_\_\_$$

4. Consider these definite integrals and how you set them up in MMA. Were your limits of integration $x = -L$ to $x = 0$, where $L = 0.30$ (in meters)? Was your integration variable dx? Was your $\vec{r} = (x_p - x)\hat{\imath} + (y_p)\hat{\jmath}$, where $x_p = 0$ and $y_p = 4.0$ (both in meters)? Did you implement correctly in MMA? etc. Working with your classmates or instructors, rework until correct. Comment in the space below on what your errors were and how you corrected them. Be specific.

PART 1: PERFECTLY GOOD FUNCTIONS WITHOUT ELEMENTARY ANTI-DERIVATIVES

5. Consider the functions $\sin\theta$, $\sin^2\theta = (\sin\theta)^2$, and $\sin\theta^2 = \sin(\theta^2)$. Which of these functions have you previously found anti-derivatives for (or equivalently, which of these functions have you integrated)? Circle them, or if you have not encountered any of these functions and their anti-derivatives, circle that choice.

$\sin\theta$	$\sin^2\theta$	$\sin\theta^2$	none of these

6. $\int \sin\theta \; d\theta$ is a standard integral, which you can find in your table of indefinite integrals (Stewart p. 358). You can evaluate $\int \sin^2\theta \; d\theta$ using the half angle formula (see Stewart p. 390, Example 2) or using integration by parts. So hopefully you circled those choices above.

7. As a class, we haven't formally encountered $\int \sin\theta^2 \; d\theta$ yet. SPOILER ALERT: it turns out that $\sin\theta^2$ does not have an elementary anti-derivative. But why not? Is there something peculiar about the graph of this function? Let's see, and along the way learn some more about MMA.

8. Launch the MMA notebook Mathematica Lab 8 Perfectly Good Functions.nb, available in Handouts: Math: Math Labs: Math Lab 8 folder. If not already, turn on Presentation mode (follow instructions on screen at start of notebook). If you haven't previously, launch the kernel by evaluating the 2+2 cell (reminder: click in the cell, then hit Shift+Enter).

9. Examine the cell labeled ML8.9. There's quite a bit going on in this cell[1]: the base function is defined, and then its variants. The three functions are also plotted. Evaluate this cell and examine the outputs. Hopefully the graphs of $\sin\theta$ and $\sin^2\theta$ look like what you expected. Is there anything unreasonable about the graph of $\sin\theta^2$, or does it look like a perfectly good function? (It is, in fact, a perfectly good function).

10. Earlier, we claimed that $\int \sin\theta \; d\theta$ and $\int \sin^2\theta \; d\theta$ could be found (and in fact, that you could find them). Let's have MMA find them. Examine the cell labeled ML8.10. Hopefully the syntax makes sense. Evaluate this cell and examine the outputs. Fill in the boxes below with the appropriate outputs[2].

$\int \sin\theta \; d\theta =$	$\int \sin^2\theta \; d\theta =$	$\int \sin\theta^2 \; d\theta$

11. The results for the first two integrals should make sense and in fact be familiar. This last one is strange. And earlier, wasn't it claimed that $\sin\theta^2$ doesn't have an elementary anti-derivative? This function, the Fresnel integral, is actually defined as $S(\theta) = \int_0^\theta \sin t^2 \; dt$, so it's not actually very useful, since we know from the FTC that if $S(\theta) = \int_0^\theta \sin t^2 \; dt$, then $S' = \frac{dS}{d\theta} = \sin\theta^2$, but this doesn't tell us what $S(\theta)$ <u>**is**</u> in terms of elementary functions. So while true, this is not useful.

12. Maybe definite integrals will work better. Examine the cell labeled ML8.12. Hopefully the syntax makes sense. Evaluate this cell and examine the outputs. Fill in the boxes below with the appropriate outputs.

$\int_0^\pi \sin\theta \; d\theta =$	$\int_0^\pi \sin^2\theta \; d\theta =$	$\int_0^\pi \sin\theta^2 \; d\theta =$

13. The first two outputs are numbers, as we expect, since a definite integral should output a number. The last output is not particularly helpful[3]. Why didn't we get a number? In cases like this, you need to tell MMA to numerically integrate the definite integral.

14. Examine the cell labeled ML8.14, which is nearly identical to the ML8.12, but instead of Integrate, we have NIntegrate. Evaluate this cell. Fill in the boxes below with the appropriate outputs. Note the ≈ instead of = signs, as numerical integrals are approximations (albeit often very good ones) for definite integrals.

$\int_0^\pi \sin\theta \; d\theta \approx$	$\int_0^\pi \sin^2\theta \; d\theta \approx$	$\int_0^\pi \sin\theta^2 \; d\theta \approx$

[1] It's not required for this lab to understand this syntax fully, but do ask at the end if you are interested in learning more.

[2] MMA doesn't include the constant of integration, but it should. (I'm not sure how to write in a purple "+ C" into MMA, but I would if I could.)

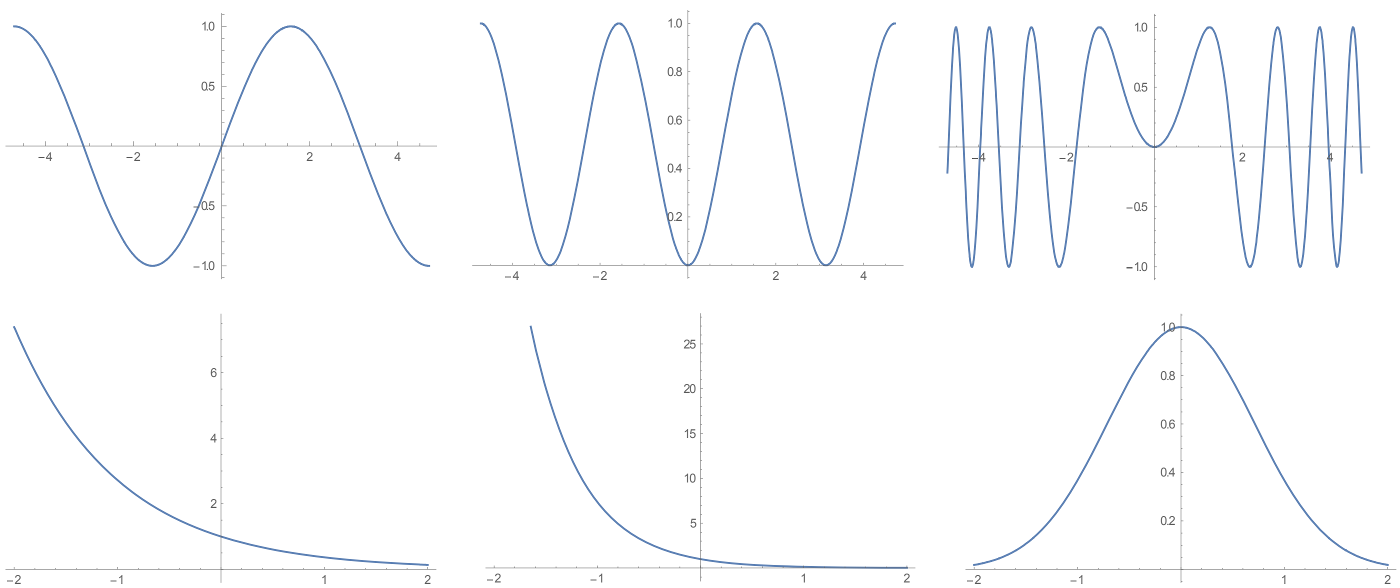
[3] Actually, this would be helpful if you had a table of output values for this special function. Such tables were painstakingly constructed using numerical methods, and filled volumes and volumes. Fun reads, fun times.

15. Repeat, but this time with the functions e^{-x}, $(e^{-x})^2 = e^{-2x}$, and e^{-x^2}. You should be able to evaluate the first two using substitution. As it turns out, (SPOILERS) e^{-x^2}, while a perfectly good function (as you will see), has no elementary anti-derivative. Examine cell ML8.15, and evaluate it. Examine the graph outputs. The first two should not be surprising. Does e^{-x^2} look like a perfectly good function?

16. Evaluate the rest of the cells, and fill out the table below[4].

$\int e^{-x}\ dx =$	$\int (e^{-x})^2\ dx =$	$\int e^{-x^2}\ dx$
$\int_{-2}^{2} e^{-x}\ dx =$	$\int_{-2}^{2} (e^{-x})^2\ dx =$	$\int_{-2}^{2} e^{-x^2}\ dx =$
$\int_{-2}^{2} e^{-x}\ dx \approx$	$\int_{-2}^{2} (e^{-x})^2\ dx \approx$	$\int_{-2}^{2} e^{-x^2}\ dx \approx$

17. For easy later reference, the graphs from this section are below. You might find it helpful to label the graphs with their function.



PART 2: SETTING UP INTEGRALS FOR THE ELECTRIC FIELD OF A UNIFORM RING OF CHARGE

18. Consider as your source a uniformly charged ring (of negligible thickness) that lies in the xy plane with its center at the origin. The ring has radius a and total charge q. Your goal is to determine the definite integrals which would allow you to calculate the x-component and y-component of the electric field at the point (x_P, y_P). Even though this ring seems two-dimensional, you can reduce it to a single variable problem by thinking about the unit circle as you've seen in both physics and math. Follow the strategy you learned in physics lecture and practiced in Math Lab 7. Don't evaluate these definite integrals just yet – only write them down by filling in the blank spaces on the next page. Use the space on the next page to show your work (attach extra pages at end if you need space).

[4] In case you are interested (it doesn't matter for this lab), Sinh is the hyperbolic sin function, defined as $\sinh x = \frac{e^x - e^{-x}}{2}$

$$E_x = k \int_{\square}^{\square} \lambda \, \frac{\rule{3cm}{0.4pt}}{\rule{3cm}{0.4pt}} \, d\_\_\_ \qquad E_y = k \int_{\square}^{\square} \lambda \, \frac{\rule{3cm}{0.4pt}}{\rule{3cm}{0.4pt}} \, d\_\_\_$$

19. Here are some intermediate pieces that were required to get to the correct integrals for this situation. For each, check the Yes box if you got it correct; otherwise, examine your work, consult neighbors and instructors as necessary, and fill in the corrected reasoning box with what your mistake was and how you now understand it.

Did you get:	Yes	If no, corrected reasoning
$\vec{r} = (x_p - a\cos\theta)\hat{\imath} + (y_p - a\sin\theta)\hat{\jmath}$	☐	
$dq = \lambda\, a\, d\theta$	☐	
$\lambda = \frac{q}{2\pi a}$	☐	
limits of integration: $\theta = 0$ to $\theta = 2\pi$	☐	

20. If necessary, use the previous step to update/correct your definite integrals below. If you are confident in your results on the previous page, go to step 21.

$$E_x = k\int_{\square}^{\square} \lambda \;\underline{\qquad\qquad\qquad}\; d\underline{\quad}$$

$$E_y = k\int_{\square}^{\square} \lambda \;\underline{\qquad\qquad\qquad}\; d\underline{\quad}$$

21. For the x-component of the electric field, you should have found

$$E_x = k\int_0^{2\pi} \lambda \frac{(x_P - a\cos\theta)}{[(x_P - a\cos\theta)^2 + (y_P - a\sin\theta)^2]^{3/2}}\, a\, d\theta$$

Did you? If yes, nice work! If not, check your work, consult with a neighbor or instructor as needed, and consider very carefully where you went wrong in step 19. and 20, and use the space below to reflect on what skills you need more practice with.

22. If necessary, also correct your E_y integral, which you should be able to do given the E_x integral in step 21.

PART 3: EVALUATING INTEGRALS, NUMERICALLY

23. Consider as your source a specific uniformly charged ring (of negligible thickness) that lies in the xy plane with its center at the origin. This ring has radius $a = 0.10$ m and total charge $q = 5$ nC. Using MMA, implement the definite integrals you determined in Part 2. I encourage you to modify the MMA notebook you used in Lab 7 to get more practice with MMA; however, I've also provided an updated one for this lab (MMA Lab 8 Single Ladies.nb[5]) if you'd rather use that or would like to check your version. Note: MMA might output some complaints, but it should also return results.

24. Confirm your work by checking the field at the center of the ring (don't forget units).

$E_x =$	$E_y =$	Does this make sense? Explain briefly.

25. Confirm your work by determining the direction of the electric field at (0, –0.50 m) (don't forget units).

$E_x =$	$E_y =$	Direction of electric field:	Does this make sense? Explain briefly.

26. Confirm your work by comparing the magnitude of the electric field at (0.30 m, 0.40 m) to your result from step 25. (don't forget units).

$E_x =$	$E_y =$	Magnitude of electric field:	Does this make sense? Explain briefly.

27. Confirm your work by determining the field at a very far distance on the x-axis, say (20.0 m, 0) (note: very far from the ring, it can be approximated as a point charge).

CHALLENGE PART 4: NON-UNIFORM CHARGE DISTRIBUTION

28. What if the charge is not uniformly distributed? Consider the ring from Part 3, but this time, have the charge density vary as a function of angle, so that $\lambda = \lambda_0 \sin\theta$, where λ_0 =1 nC/m. How does your integral change? How do you implement this in MMA? Report the electric field at the center of the ring, at (0.30m, 0.40 m), and far from the ring. Comment on your results. Attach extra pages as needed.

[5] 'Cause if you like it then you should have put a ring on it