Math Lab 7:
Electric Fields – Integrating Continuous Charge Distributions I
Due noon Thu. Jan. 25 in class

Goals:
1. More practice with calculating electric fields, particularly with finding the vector from source points to field point and breaking vectors into components.
2. Practice setting up integrals to calculate the electric field due to continuous charge distributions.
3. Practice techniques of integration.
4. Learn to use *Mathematica* to evaluate indefinite and definite integrals.

Instructions: Same as previous labs; see Handout folder in program file share.

INTRODUCTION

Calculating the electric field at an arbitrary field point due to some continuous distribution of charge has many elements in common with calculating the electric field from discrete point charges. All these types of problems have the same general strategy, which should look familiar to you from what you saw in physics lecture. A very detailed, step-by-step strategy (with recommendations) is provided below – you will get practice deploying this strategy in this lab and next week.

Strategy for Calculating Electric Field due to Continuous Charge Distributions

a) Sketch the situation, showing the charge distribution and coordinate system. (If you are able to choose the coordinate system, it's useful to choose one that takes advantage of any symmetries of the charge distribution; for example, if you have a line charge, draw the coordinate system so the line charge lies along one of the axes.)
b) Split the object into some number of small pieces, dq. Treat each piece as a point charge.
c) Pick an arbitrary piece of charge, and determine its coordinates as generally as possible. (Avoid picking "special pieces", such as end pieces or middle pieces.) These coordinates will often just be (x, y); if one of the coordinates is a constant value (for example, you have a line charge along the x-axis so all the y-values are 0) it is often useful to use this information at this step.
d) Identify the field point where the field is to be calculated (and if possible, sketch it), with general coordinates (x_P, y_P). (Even if given a specific point, it's often useful to wait till the end to plug in particular values of x_P and y_P, unless one or both are zero.)
e) Determine $\vec{r}$, the vector which points FROM the charge TO the field point, using $\vec{r} = (x_P - x)\hat{\imath} + (y_P - y)\hat{\jmath}$ and simplifying using decisions you made in steps c) and d). Use $\vec{r}$ to find $r = |\vec{r}| = \sqrt{(x_P - x)^2 + (y_P - y)^2} = [(x_P - x)^2 + (y_P - y)^2]^{1/2}$.
f) Find the electric field $d\vec{E}$ due to the arbitrary piece of charge dq, using
$$d\vec{E} = k\frac{dq}{r^2}\hat{r} = k\frac{dq}{r^2}\frac{\vec{r}}{r} = k\,dq\,\frac{\vec{r}}{r^3} = k\,dq\,\frac{(x_P - x)\hat{\imath} + (y_P - y)\hat{\jmath}}{\left([(x_P - x)^2 + (y_P - y)^2]^{\frac{1}{2}}\right)^3}$$
g) Break $d\vec{E}$ into components dE_x and dE_y:
$$dE_x = k\,dq\,\frac{(x_P - x)}{[(x_P - x)^2 + (y_P - y)^2]^{3/2}}$$
$$dE_y = k\,dq\,\frac{(y_P - y)}{[(x_P - x)^2 + (y_P - y)^2]^{3/2}}$$
h) Determine dq in terms of geometry and/or coordinates of source points and use this to determine the integration variable and limits of integration. (This is usually one of the harder parts of these types of problems.) Substitute this in to your dE_x and dE_y expressions.
i) Integrate dE_x to find $E_x = \int dE_x$ and dE_y to find $E_y = \int dE_y$ using techniques of integration, numerical methods, and/or technology. (If you had a particular field point (x_P, y_P), substitute for it now if you haven't already.)
j) ASSESS that your answer makes sense, checking directions, units, limiting cases, etc.

PART 1: SETTING UP INTEGRALS FOR THE ELECTRIC FIELD OF A FINITE ROD OF UNIFORM CHARGE

To practice this strategy, you will find the electric field for a rod of uniformly distributed charge anywhere around the rod. The special case of the observation point being along the perpendicular bisector is done exactly in the text (see Example 20-7, where it is done for an very long (infinite) rod but the procedure would be the same for a finite rod except for the limits of integration, and Problem 20.72 describes this situation), but the general problem (see Problem 20.78, which is just like this problem) is harder.

Consider a rod of length L that lies along the y-axis, with its center at the origin, with charge q uniformly distributed along its length. Your goal is to determine the definite integrals which would allow you to calculate the x-component and y-component of the electric field at the point (x_P, y_P). Don't evaluate these definite integrals just yet – only write them down by filling in the blank spaces below.

1. If you think you can follow the strategy described above and/or are feeling up for a challenge, proceed using the space below to show your work (attach extra pages at end if you need space). If you'd like some support, go to Handouts: Math: Math Labs: Math Lab 7 folder, and open up the PowerPoint file Math Lab 7 Tutorial, and also use the space below to record your work (attach extra pages at end if you need space).

$$E_x = k\int_{\square}^{\square} \frac{\quad}{\quad} \, d\_\_\_ \qquad\qquad E_y = k\int_{\square}^{\square} \frac{\quad}{\quad} \, d\_\_\_$$

PART 2: EVALUATING INTEGRALS, ANALYTICALLY

2. For the y-component of the electric field, you should have found

$$E_y = k\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{q}{L}\frac{(y_P - y)}{[(x_P)^2 + (y_P - y)^2]^{3/2}}\,dy = k\frac{q}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(y_P - y)}{[(x_P)^2 + (y_P - y)^2]^{3/2}}\,dy$$

Did you? If not, check your work, and consult with a neighbor or instructor as needed.

3. While the (indefinite) integral $\int \frac{(y_P-y)}{[(x_P)^2+(y_P-y)^2]^{3/2}}\,dy$ certainly looks intimidating, we can actually integrate this using techniques we encountered earlier this quarter. Look closely at the integrand and see if you can figure out how to proceed. If you know how, evaluate this indefinite integral, showing work in the space below. u can find a hint at front table.

4. You should have found that $\int \frac{(y_P-y)}{[(x_P)^2+(y_P-y)^2]^{3/2}}\,dy = [(x_P)^2 + (y_P - y)^2]^{-1/2}$.

Then, evaluating the definite integral $k\frac{q}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(y_P-y)}{[(x_P)^2+(y_P-y)^2]^{3/2}}\,dy$ would give

$$\left[(x_P)^2 + \left(y_P - \frac{L}{2}\right)^2\right]^{-1/2} - \left[(x_P)^2 + \left(y_P - \left(-\frac{L}{2}\right)\right)^2\right]^{-\frac{1}{2}} = \left[(x_P)^2 + \left(y_P - \frac{L}{2}\right)^2\right]^{-1/2} - \left[(x_P)^2 + \left(y_P + \frac{L}{2}\right)^2\right]^{-\frac{1}{2}}$$

How will you know if you are right? Well, you could check with a neighbor. Or you could take a derivative. Or, you could check a special case, which you will do next.

5. Consider a point on the perpendicular bisector, which in this case would be on the x-axis. Then, the field point (x_P, y_P) would be $(x_P, 0)$. Looking at your diagram and considering symmetry, what should E_y, the y-component of the electric field, be on the perpendicular bisector? If you substitute $y_P = 0$ into $\left[(x_P)^2 + \left(y_P - \frac{L}{2}\right)^2\right]^{-1/2} - \left[(x_P)^2 + \left(y_P + \frac{L}{2}\right)^2\right]^{-\frac{1}{2}}$, what do you get?

6. For the x-component of the electric field, you should have found $E_x = k\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{q}{L} \frac{(x_P)}{[(x_P)^2+(y_P-y)^2]^{3/2}} dy =$

$k\frac{q}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(x_P)}{[(x_P)^2+(y_P-y)^2]^{3/2}} dy$. Did you? If not, check your work, and consult with a neighbor or instructor as needed. This integral is technically evaluable using a combination of u-substitutions and trig substitutions; BUT, we won't do that. Your physics book suggests using a table of integrals, and in fact it is in your calculus book on Reference Page 6 in the back, equation number 29. BUT, we won't do that either. Instead, we will use the symbolic computation program *Mathematica*.

PART 3: EVALUATING INTEGRALS, *Mathematica*-lly

7. *Mathematica* (MMA) is extremely powerful, but has a somewhat steep learning curve and is quite particular about its syntax. It's also expensive, but we currently have a license for use on campus. It's worth the time to learn it at some point, but for today, you'll make modifications to shells given to you. You can find the shells in Handouts: Math: Math Labs: Math Lab 7 folder; launch Mathematica Lab 7Getting Started.nb.

8. Commands are entered into what are called "cells"; you evaluate a cell by clicking in it and then pressing Shift+Enter (press the Shift key and the Enter key at the same time) or sometimes by pressing Enter on the numeric keypad. For future reference, MMA has an extremely detailed Help files, which you can find using the Help menu and clicking either on Wolfram Documentation or Find Selected Function.

9. The first cell has 2 + 2. Click in that cell, then evaluate it by pressing Shift+Enter as described above. This will "initialize the kernel"; the kernel is (kind of) where the calculations take place and so we don't want to launch the kernel with something complicated that might have a mistake in it. When it's done, you will see some new blue text added just before the 2 + 2, and you should also see the output 4. The output of the cell might be small; you can enlarge it by going to the Format menu, Screen Environment, then choosing Presentation (there are probably other ways also).

10. The next cell should look familiar to you from 3. in Part 2. There is an indefinite integral template in the lower left hand corner of the Basic Math Assistant. If you don't have the Basic Math Assistant on your screen and you want it, you can go to the Palette menu, where you can choose it. The Basic Math Assistant also has templates for superscript and fractions (and if you hover over the template button, you see the keyboard shortcut). Click in this cell and evaluate it. Compare the output to your result from 3. and 4. in Part 2. Good?

11. The next cell is the same content as the previous cell, but typed out rather than using the templates. Hopefully the terms make sense to you. Ask if you have any questions. Evaluate this cell. The output should hopefully not be surprising.

12. Either by using the Basic Math Assistant templates or by modifying what is there, enter $\int \frac{(xP)}{[(xP)^2+(yP-y)^2]^{3/2}} dy$ into your MMA notebook, and evaluate the cell. Write down your result:

$$\int \frac{(xP)}{[(xP)^2+(yP-y)^2]^{3/2}} dy =$$

13. Either by modifying what is there or by entering it yourself, input the typed out syntax for the above, evaluate the cell, and see what you get. Again, the output should match.

PART 4: INTEGRATING ALL THE PIECES

14. We haven't actually calculated any electric fields yet. We need to evaluate some <u>definite</u> integrals, and we have k, q, and L to include. From the Math Lab 7 folder, launch MMA Integrating All The Pieces.nb, and look it over. Hopefully most if not all the parts make sense; if not, please ask. In this notebook, we have a 1 meter long rod with 10 μC of charge uniformly distributed over its length, along the y-axis with its center at the origin. The field point is (0.3 m, 0.4 m).

15. Evaluate the main cell; the output for E_x should be 379473 and for E_y should be 189737. This is a ridiculous amount of digits to report, but ok. What are the units?

16. Set the field point to (–0.5 m, 0 m). What is E_y? Does this make sense? Why?

$E_y =$	make sense?

17. Now, consider a 0.30 m long rod with 6 nC of charge uniformly distributed over its length. The rod lies along the x-axis, with its right end at the origin. Determine E_x and E_y at the field point (0 m, 4 m) by following the strategy to find the definite integrals and modifying the MMA notebook to evaluate the definite integrals.

$E_x =$

$E_y =$