Math Lab 6: Powerful Fun with Power Series Representations of Functions
Due noon Thu. Jan. 11 in class (*note new due time, location for winter quarter)

Goals:
1. Practice taking derivatives.
2. Investigate how to approximate arbitrary functions using power series representations.
3. Determine Taylor series representations of common functions.

Instructions: Same as previous labs; see Handout folder in program file share.

INTRODUCTION

In this lab, you will learn how to approximate an arbitrary function with a power series (polynomial) representation (and start to learn what a power series is). We will return to this in spring quarter where we will build a more solid theoretical foundation, but it's very useful in the physical sciences to be able to replace more complicated functions with simpler power series representations. An example would be trying to integrate a function that does not have an elementary anti-derivative: replace the function with its power series representation, which can be integrated (you can always find the anti-derivative of a power function) and you have an approximate integral for your original function. You've also encountered versions of this in physics; for example, approximating $\sin x \approx x$ for "small" x is a power series representation, since $x = x^1$ is a power function. Another place this showed up in the background was using Euler's method to numerically solve the differential equation for a spring-mass system or the simple pendulum.

PART 1: LINEAR APPROXIMATIONS TO FUNCTIONS

1. Use Desmos to plot $f(x) = (x+1)^{1/2}$. In the next input box, enter (0, 1) to plot that point. Zoom in on the point until the curve looks linear.
2. In the space below, show that the equation to the tangent line to $f(x) = (x+1)^{1/2}$ at the point (0, 1) is $y = \frac{1}{2}x + 1$.

3. On the same graph (still in default zoom), plot $g(x) = \frac{1}{2}x + 1$. What do you notice about the curve and the tangent line to the curve? Hopefully this reminds you that the tangent line to a curve at a point is the local linearization of the curve.
4. Note the range and domain of the plot window for which the tangent line is an excellent approximation to the function. Try to have the point (0, 1) as centered on your screen as you can.

$\_\_\_\_\_\_\_ \le x = 0 \le \_\_\_\_\_\_\_ \qquad \_\_\_\_\_\_\_ \le y = 1 \le \_\_\_\_\_\_\_$

5. Now, zoom out until you can see clear deviations between the tangent line and the function. Again, try to have the point (0, 1) as centered on your screen as you can. Note the range and domain of the plot window.

$\_\_\_\_\_\_\_ \le x = 0 \le \_\_\_\_\_\_\_ \qquad \_\_\_\_\_\_\_ \le y = 1 \le \_\_\_\_\_\_\_$

PART 2: QUADRATIC APPROXIMATIONS TO FUNCTIONS

6. Let's examine the equation $g(x) = \frac{1}{2}x + 1$ further. We can re-write this as $g(x) = 1 \cdot x^0 + \frac{1}{2} \cdot x^1$. In the space below, show that you understand this move.

7. The expression $y = 1 \cdot x^0 + \frac{1}{2} \cdot x^1$ is made up of power functions with a term of degree 0 (the x^0 term) and term of degree 1 (the x^1 term). To anticipate something we'll need later, we can generalize this to $g(x) = c_0 x^0 + c_1 x^1 = c_0 + c_1 x$, where c_0 and c_1 are constants.

In this case $g(x) = 1 + \frac{1}{2}x$, what is c_0? What is c_1? $c_0 =$ ____________ $c_1 =$ ____________

8. Let's add another power function to the two we already have, so we can go from a linear function to a quadratic function. In Desmos, type in h(x) = 1 + 1/2 * x + c_2*x^2 which should display as $h(x) = 1 + \frac{1}{2}x + c_2 x^2$. Add in a slider for c_2. Set the range for the slider to be between -1 and 1.

9. Set $c_2 = 0$ initially. For this case, the line for $g(x)$ and the graph for $h(x)$ should be identical. Adjust c_2 until the graph of $h(x)$ is a better match to the graph of $f(x)$ than the line $g(x)$ is. What is your best value for c_2?

$c_2 =$ ____________

11. Was your answer for c_2 close to $-\frac{1}{8} = -0.125$? If not, try again or check in with a neighbor or instructor.

12. We can conclude that the quadratic function $h(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ is a better approximation to $f(x) = (x+1)^{1/2}$ than the linear function $g(x) = 1 + \frac{1}{2}x$, at least near the point (0, 1). We won't do it for this part, but we could add a cubic term with some constant, a degree 4 terms with another constant, etc. and it might not surprise us if we got better and better fits to the original function.

13. Let's see how good our power function approximations are. Let's consider $x = 0.2$. BY HAND (please don't use your calculator), calculate $g(0.2)$:

$g(0.2) = 1 + \dfrac{1}{2}(0.2) =$

14. Again BY HAND (please don't use your calculator), calculate $h(0.2)$; you should get 1.095:

$h(0.2) = 1 + \dfrac{1}{2}(0.2) - \dfrac{1}{8}(0.2)^2 =$

15. Now, please DO USE your calculator to calculate $f(0.2)$:

$f(x) = (0.2 + 1)^{1/2} = (1.2)^{1/2} =$

16. Compare your answers. Is the cubic polynomial $h(x)$ a better approximation to the function $f(x)$ than the quadratic polynomial $g(x)$ at $x = 0.2$? What about on its own terms: is $h(0.2)$ close to $f(0.2)$?

17. Go back to Desmos, and return to the default zoom (use the  button). Do you think $g(2)$ or $h(2)$ would give good approximations to $f(2)$?

18. BY HAND, calculate $g(0.02)$	BY HAND, calculate $h(0.02)$	Use a calculator to determine $f(0.02)$

19. Compare your answers. Is the cubic polynomial $h(x)$ a **<u>much better</u>** approximation to the function $f(x)$ than the quadratic polynomial $g(x)$ at $x = 0.02$?

20. Now, also consider the ease of calculating $g(0.02)$ compared to calculating $h(0.02)$. Which was easier to calculate? What would you say to the claim "$(x+1)^{1/2} \approx 1 + \frac{1}{2}x$ for 'small enough' x"?

PART 3: CUBIC POLYNOMIAL APPROXIMATION TO THE EXPONENTIAL FUNCTION NEAR $x = 0$

21. We will proceed similarly to before, but this time for $f(x) = e^x$ near $x = 0$. As before, we'll start with the linear approximation by finding the equation of the tangent line at $x = 0$. In the space below, show that the equation to the tangent line to $f(x) = e^x$ at the point $(0, 1)$ is $g(x) = 1 + x$.

22. In a new browser tab or window, open https://www.desmos.com/calculator/ipwbcpyvh5. To increase ease in viewing, turn on Projector Mode under graph settings (use the wrench). This Desmos calculator plots $f(x) = e^x$ and the point $(0, 1)$. You will also see expressions for $g(x)$, $h(x)$, and $i(x)$. Initially, the constants c_0, c_1, c_2, and c_3 are all set to zero. Consider the expression for $g(x)$ you confirmed in step 21.
In the Desmos calculator, $g(x) = c_0 + c_1x$. What are the numerical values for c_0 and c_1? $c_0 =$ ______ $c_1 =$ ______

23. In the Desmos calculator, set $c_0 = 1$ and $c_1 = 1$. The graph for $g(x)$ should appear.

24. Turn on the graph for $h(x)$ by clicking on the circle in input box 4, next to the expression for $h(x)$. Since c_2 is currently set to 0, $h(x) = g(x)$ so the two graphs should be identical. So, what is c_2? As you did before, you could just guess at different values for c_2 using the slider until it looks "good enough". It turns out that if there exists a power series representation for a function over some interval, then there is a formula to calculate the various c_n constants for each x^n term.

25. For a function $f(x)$ that can be represented by a power series over some interval, then in a region near $x = a$:

$$f(x) = c_0(x-a)^0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

Since $(x-a)^0 = 1$ and $(x-a)^1 = (x-a)$, then:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

Including more and more terms in the power series results in a better approximation over a larger range interval.

The c coefficients are given by:

$$c_0 = \frac{f(a)}{1} \qquad c_1 = \frac{f'(a)}{1} \qquad c_2 = \frac{f''(a)}{1 \cdot 2} \qquad c_3 = \frac{f'''(a)}{1 \cdot 2 \cdot 3} \qquad c_4 = \frac{f^{(iv)}(a)}{1 \cdot 2 \cdot 3 \cdot 4} \qquad \cdots \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$ and $n!$ (pronounced "n factorial") is defined such that $0! = 1, 1! = 1, 2! = 1 \cdot 2, 3! = 1 \cdot 2 \cdot 3, 4! = 1 \cdot 2 \cdot 3 \cdot 4, \ldots, n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$

We'll see where the formula for the coefficients comes from in Chapter 8 at the beginning of spring quarter, but if you can't wait until then, you can glance over section 8.7. We won't worry about how to derive this formula now, but we'll practice using it and see that it works. These types of series are called Taylor Series, and for the special case where $a = 0$, these are called MacLaurin Series.

26. Fill out the following table; the first few rows and some other entries are done for you to follow as an example. Recall that in our case, $f(x) = e^x$, expanded around $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$	$n!$	$c_n = \frac{f^{(n)}(a)}{n!}$
0	$f(x) = e^x$	$f(0) = e^0 = 1$	$0! = 1$	$c_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$
1	$f'(x) = e^x$	$f'(0) = e^0 = 1$	$1! = 1$	$c_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1$
2	$f''(x) =$	$f''(0) =$	$2! =$	$c_2 = \frac{f''(0)}{2!} =$
3	$f'''(x) =$	$f'''(0) =$	$3! =$	$c_3 = \frac{f'''(0)}{3!} =$

27. Note that $c_0 = 1$ and $c_1 = 1$, which is the same thing found in step 22, where they were obtained from the equation for the tangent line to $f(x) = e^x$ at $x = 0$. You should have found that $c_2 = 1/2$ and $c_3 = 1/6$. Did you? If not, try again or check in with a neighbor or instructor. In fact, you may be able to guess that the general formula for $c_n = 1/n!$.

28. Go back to the Desmos calculator, and set $c_2 = 1/2$. What do you notice about the match between the quadratic polynomial $h(x)$ and the function $f(x)$ near $x = 0$, especially compared with the linear $g(x)$?

29. Now, turn on the graph for $i(x)$ by clicking on the circle in input box 5, next to the expression for $i(x)$. Since c_3 is currently set to 0, $i(x) = h(x)$ so the two graphs should be identical. Set $c_3 = 1/6$. What do you notice about the match between the cubic polynomial $i(x)$ and the function $f(x)$ near $x = 0$, especially compared with the linear $g(x)$ and the quadratic $h(x)$?

30. Turn off $g(x), h(x)$, and $i(x)$ by clicking on the circles next to the input boxes. Go back to the default zoom (use the  button). One at a time, turn on $g(x), h(x)$, and $i(x)$ by clicking on the circles.

31. Briefly summarize what you learned in this part.

PART 4: POWER SERIES FOR SIN NEAR $x = 0$

32. In a new browser tab or window, open https://www.desmos.com/calculator/epwg13gztm. To increase ease in viewing, turn on Projector Mode under graph settings (use the wrench). This calculator graphs $f(x) = \sin x$. There is also a $g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$ which is currently turned off, with all the coefficients set to zero.

33. Fill out the following table to find the first terms of the power series representation for $f(x) = \sin x$, expanded around $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$	$n!$	$c_n = \frac{f^{(n)}(a)}{n!}$
0	$f(x) = \sin x$	$f(0) = \sin 0 = 0$	$0! =$	$c_0 = \frac{f(0)}{0!} = \frac{0}{1} = 0$
1	$f'(x) =$	$f'(0) =$	$1! =$	$c_1 = \frac{f'(0)}{1!} =$
2	$f''(x) =$	$f''(0) =$	$2! =$	$c_2 = \frac{f''(0)}{2!} =$
3	$f'''(x) =$	$f'''(0) =$	$3! =$	$c_3 = \frac{f'''(0)}{3!} =$
4	$f^{(4)}(x) =$	$f^{(4)}(0) =$	$4! =$	$c_4 = \frac{f^{(4)}(0)}{4!} =$
5	$f^{(5)}(x) =$	$f^{(5)}(0) =$	$5! =$	$c_5 = \frac{f^{(5)}(0)}{5!} =$

34. Did you find $c_0 = c_2 = c_4 = 0$ and $c_1 = 1$, $c_3 = -1/6$, and $c_5 = 1/120$? If not, try again or check in with a neighbor or instructor.

35. Turn on $g(x)$ by clicking on the circle next to input box 2. One at a time in order, set c_0, c_1, c_2, c_3, c_4, and c_5 to the calculated values. As you set $c_1 = 1$, note that this is saying that $\sin x \approx x$ for small x. What do you notice about the polynomial approximation $g(x)$ to the function $f(x) = \sin x$ as you add more terms?

PART 5: APPROXIMATING $(1 + x)^k$ for small x

36. Previously, we found that $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$ for "small enough" x. We also found that $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ was better for not quite as small x. But for small enough x, we are able to neglect x^2 and higher order x terms (e.g. x^3, x^4, etc.) because if x is close to 0, then x^2 is even smaller. Another way to think of this, as we've seen, is that this is a valid approximation when the tangent line is a good approximation to the function itself. We can determine the approximation either by finding the tangent line or by finding the first few c_n coefficients as we did in the last two parts. Let's try a few and see if we can find a pattern.

37. Try $(1 + x)^2$. Either by finding the tangent line to $(1 + x)^2$ at $x = 0$ or by finding c_0 and c_1, show that $(1 + x)^2 \approx 1 + 2x$ for small x.

38. Another way to show this is by expanding the binomial $(1+x)^2$. Show that expanding $(1+x)^2 = 1 + 2x + x^2$, so that for small x, we again get that $(1+x)^2 \approx 1 + 2x$.

39. Try for $(1+x)^{1/3}$, either by finding the tangent line at $x = 0$ or by finding c_0 and c_1.

40. Try for $(1+x)^{-1/2}$.

41. Summarizing these in the table, we find that (for small x):

$k = 1/2$	$(1+x)^{1/2} \approx 1 + \frac{1}{2}x$
$k = 2$	$(1+x)^2 \approx 1 + 2x$
$k = 1/3$	$(1+x)^{1/3} \approx 1 + \frac{1}{3}x$
$k = -1/2$	$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$

Make a conjecture for $(1+x)^k$, where k is any number.

42. Prove your conjecture: