Matter & Motion – Fall 2017-18 Name:______________________________

Partners' names (if any):______________________Partners' contact info:______________________________

Estimated time spent on this lab (circle best one): less than 2 hours between 2 and 3 hours between 3 and 4 hours more than 4 hours

Math Lab 2: Learning About Limits and Discovering the Derivative[1]

Due 5 pm Thu. Oct. 5 to box outside Lab 2 3255

Goals:

1. Gain more familiarity with Desmos and/or Excel.
2. Develop an intuitive understanding of the nature of limits to lay the foundation for their frequent use in calculus.
3. Experience the power and perils of investigating limits by successively closer evaluation.
4. Calculate average velocities and see how a limiting procedure connects average velocity over an interval to instantaneous velocity at an instant.
5. Review and practice with the 2 point version for the equation of a line.
6. Use Desmos to draw secant lines for arbitrary functions.
7. See how the tangent line is the limit of the secant line
8. Describe the derivative graphically as the slope of the tangent line to the graph of a function.
9. Think meta-cognitively about your learning in this activity.

Instructions:

- Work on a CAL computer or your own laptop. If using your own laptop, don't take up a seat at a CAL computer so one of your classmates can use that computer instead.
- Work by yourself or with a partner. If working with a partner, make sure to trade the keyboard frequently, since one of the Lab Learning Goals is to gain more familiarity with Desmos and/or Excel.
- There are many repetitive calculations in these exercises. The instructions below assume you are using Desmos. You may prefer to use Excel instead, which is fine.
- Complete the exercises and respond in the appropriate entry boxes. If you need more space, type out or neatly write out your responses on separate pages, number the pages, and staple them in order to the Lab Write-up; in the entry box, please note that your response is continued and indicate the page number.
- A program tutor is available right after the Math Lab session, from 1-3pm, in case you want to finish the Math Lab right away. The program tutor is also available to support your learning in all parts of the program.

PART 1: LEARNING ABOUT LIMITS

In this part, you will study the behavior of a function near a specified point. While this is sometimes a straightforward process, it can also be quite subtle; in many instance in calculus, the process for finding a limit must be applied carefully. By gaining an intuitive feel for the notion of limits, we will be laying a solid foundation for success in calculus.

1. Consider the function f defined by $f(x) = \frac{x^4-1}{x-1}$ near $x = 1$.

a) Fill out the table below using Desmos or Excel. Desmos instructions are below.

b) Launch Desmos. Enter f(x) = (x^4 – 1)/(x – 1) and inspect the result to make sure you have it entered correctly. As you've seen, a graph will automatically appear. You can evaluate various inputs just by using $f(x)$. For example, on a new line, enter f(1.8); Desmos will automatically display the function output. You can just keep changing the input on that line and writing down the new output.

x	1.8	1.9	1.99	1.999	1.9999
$f(x)$					

[1]These exercises are based on Lab 2: Introduction to Limits of Functions and Lab 4: Discovering the Derivative, in *Learning by Discovery: A Lab Manual for Calculus*, ed. Anita Solow, 1993, MAA.

c) Keep adding 9's so that you have 1.99999, 1.999999, 1.9999999, etc. until the output doesn't change anymore. What's that unchanging output? What do you notice happens to the values of $f(x)$ as x increases towards 2?

d) Repeat, but this time fill out the table below:

x	2.2	2.1	2.01	2.001	2.0001
$f(x)$					

e) Keep adding 0's so that you have 2.00001, 2.000001, 2.0000001, etc. until the output doesn't change anymore. What's that unchanging output? What do you notice happens to the values of $f(x)$ as x decreases towards 2?

f) We describe what you found in b) and c) by writing $\lim_{x\to 2^-} f(x) = 15$, and what you found in d) and e) as $\lim_{x\to 2^+} f(x) = 15$. What is $\lim_{x\to 2} f(x)$?

g) Note that in the tables above, you never actually found $f(2)$. Input f(2) into Desmos. What is $f(2)$?

h) Is $\lim_{x\to 2} f(x) = f(2)$? **YES** **NO**

2. The above might not have been very interesting nor particularly revealing. Instead, consider what happens as $x \to 1$.
a) Fill out the table below.

x	0.8	0.9	0.99	0.999	0.9999
$f(x)$					

b) Keep adding 9's so that you have 0.99999, 0.999999, 0.9999999, etc. until the output doesn't change anymore. What's that unchanging output? What do you notice happens to the values of $f(x)$ as x increases towards 1?

c) Repeat, but this time fill out the table below:

x	1.2	1.1	1.01	1.001	1.0001
$f(x)$					

d) Keep adding 0's so that you have 1.00001, 1.000001, 1.0000001, etc. until the output doesn't change anymore. What's that unchanging output? What do you notice happens to the values of $f(x)$ as x decreases towards 1?

e) What is $\lim_{x\to 1^-} f(x)$? What is $\lim_{x\to 1^+} f(x)$? What is $\lim_{x\to 1} f(x)$?

f) Note that in the tables above, you never actually found $f(1)$. Input f(1) into Desmos. What is $f(1)$? Why?

g) Is $\lim_{x\to 1} f(x) = f(1)$? **YES** **NO**

Note: you did this numerically. You could also have done it graphically, by dragging your mouse along the graph near the two points of interest. It will help to zoom in the graph if you choose to do this.

There are situations in which direct evaluation at the specified point is possible and actually gives the limit, as you saw when exploring near $x = 2$. These give rise to a concept called *continuity*. There are many important situations in calculus when this technique will not work, however, as you saw near $x = 1$.

3. Using graphing (or function evaluation at very nearby points), try to determine the values of the following limits, and write down any useful notes or observations.

a) $\lim_{x\to 0} \frac{\sin(10x)}{x}$

b) $\lim_{x\to 0}(1+x)^{1/x}$.
Record your best guess; the limit is a famous mathematical constant

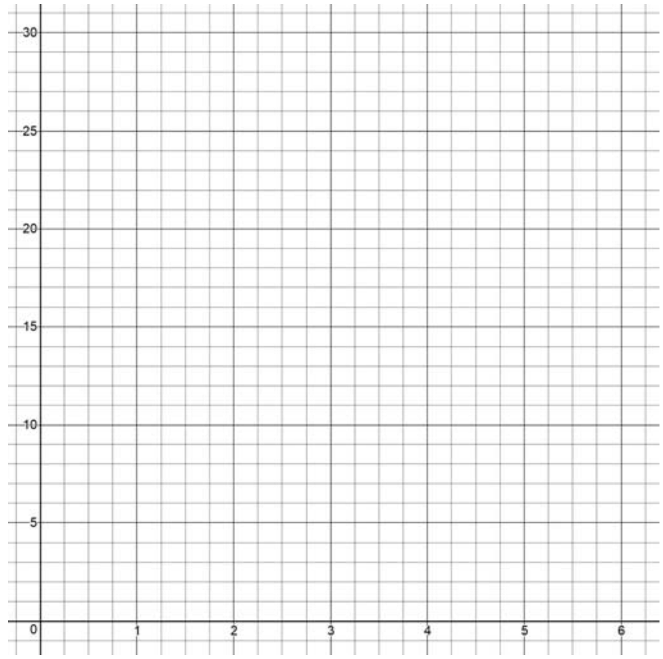
c) $\lim_{x\to 1} \frac{x^n-1}{x-1}$, for a general positive integer n. Hint: test various values of n (use a slider for n, and set the step size to be 1; ask if you're not sure how), then generalize. Also notice what happens to the shape of the graph as you change n in integer steps.

4. It is important to be aware that (finite) limits can sometimes fail to exist. Investigate the following limits and explain why you think each does not exist; a graph or a sketch of a graph will go a long way to helping you explain.

a) $\lim_{x\to 2} \frac{x}{x-2}$	c) $\lim_{x\to 0} \sin\frac{1}{x}$
b) $\lim_{x\to 0} \frac{\sin(10x)}{x^2}$	d) $\lim_{x\to 0} \frac{\lvert x\rvert}{x}$

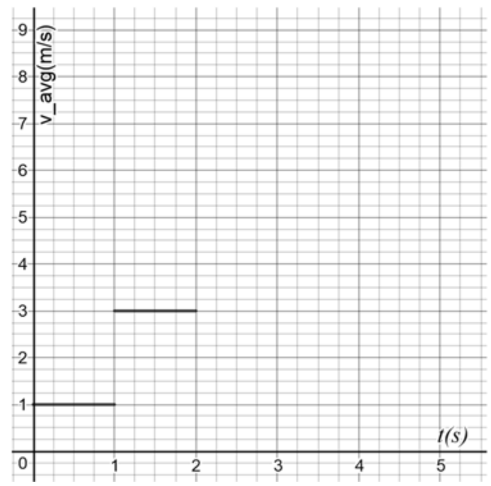
PART 2: VELOCITY AT AN INSTANT

1. Consider the following table representing the time t (s) and position x (m) of some object. **Draw the associated position vs. time graph on the axes provided (label axes).**



t (s)	x (m)	Δt (s)	Δx (m)	$v_{avg} = \Delta x/\Delta t$ (m/s)
0	1			
		1 – 0 = 1	2 – 1 = 1	1/1 = 1
1	2			
		2 – 1 = 1	5 – 2 = 3	3/1 = 3
2	5			
3	10			
4	17			
5	26			

2. Calculate the change in time during each of the first 5 seconds (recall: $\Delta t = t_f - t_i$, and all the time steps are equal in this case) and the change in position during each of the first 5 seconds (straightforward, but remember to subtract the position at the beginning of the interval from the position at the end of the interval: $\Delta x = x_f - x_i$). You'll see the first few entries are done for you, **fill out the rest of the Δt and Δx columns in the table.** Note the layout of the table, which encourages you to recognize that these quantities are associated with intervals and not with instants.



3. What trend do you notice in the Δx column?

4. Next, calculate the average velocities during each of the first 5 seconds, and fill out the appropriate column on the table; two entries are done for you.

5. On the axes provided, draw a graph of average velocity vs. time. Note that the graph is partially drawn for you, again indicating that we associated average velocity with an interval, not an instant.

6. Consider $t = 2$s. What could you report as "the" velocity at $t = 2$s? Describe the difficulty in answering this question. Brainstorm what might be some good ways to approximate "the" velocity at $t = 2$s.

7. The values in the above table showed some points for the function $x(t) = at^2 + b$, where $a = 1$ m/s^2 and $b = 1$ m. Fill out the entries in the two tables on the next page. Note that each table use smaller and smaller time steps.

t (s)	x (m)	Δt (s)	Δx (m)	$v_{avg} = \frac{\Delta x}{\Delta t}$ (m/s)
1.50	3.25			
		0.25		
1.75	4.0625			
		0.25		
2.00	5			
		0.25		
2.25	6.0625			
		0.25		
2.50	7.25			

t (s)	x (m)	Δt (s)	Δx (m)	$v_{avg} = \frac{\Delta x}{\Delta t}$ (m/s)
1.80	4.24			
		0.10		
1.90	4.61			
		0.10		
2.00	5			
		0.10		
2.10	5.41			
		0.10		
2.20	5.84			

8. Can you make a more confident claim as to what “the” velocity at $t = 2$s might be? Briefly explain.

PART 3: SEEKING SECANTS

1. In a blank Desmos calculator, enter f(x) = x^2.

2. Make a movable point by entering the ordered pair (a, b) in a new line. Turn on the requested sliders. You can either use the sliders to move the point, or equally fun, actually move the point directly using the mouse.

3. Note that this point is not connected to the graph of f(x) = x^2 in any way. In order to have the movable point be on the graph of f(x), change the ordered pair (a, b) to the ordered pair (a, f(a)). Get rid of the b slider since it’s just clutter at this point. Now move the point around and verify that it falls on and follows the graph of f(x).

4. Make a second movable point (a+h, f(a+h) in a new line. Turn on the slider requested for h. Verify that you now have two points that lie on the graph of f(x), which you can move around independently.

5. The default values for a (look on either side of the a slider) are probably sufficient though you might find you want to extend the values. For later parts, you will find it convenient to change the default values for h. Set the lowest value for h be 0.001 and the upper bound to be 5 (feel free to change this upper (and lower) bound as needed).

6. You may also find it convenient later on to have the output for any input a shown. Enter f(a) on a new line, which should do this.

7. Your goal is to get a line that connects the two points (a, f(a)) and (a+h, f(a+h)). For the two points (a, f(a)) and (a+h, f(a+h)), write down a formula for the slope m in terms of any of a, h, a+h, f(a), f(a+h), etc. Explain your reasoning as needed. If you’re stuck, peek ahead to the next step.

8. Enter m = (f(a+h) - f(a))/h into Desmos.

9. If you got a different expression for m on your own in step 7., explain how the form in step 8. is consistent with your form as needed. If you were not able to get an expression for m on your own, take the time to write down a clear explanation for why this expression gives you the slope.

10. You now have two points (a, f(a)) and (a+h, f(a+h)) as well as an expression for the slope m. Determine an equation for the line passing through these two points, in terms of the relevant variables. If you’re not sure, that’s ok – consult with a neighbor, tutor, or faculty.

11. Enter this expression into Desmos. If you have done this correctly, you will have a line that goes through the two points, and **you should be able to adjust this line by moving the points around.** If this works, great. If you can't get it, that's ok – check with a neighbor, tutor, or faculty.

You should now have a line through two points on a graph of f(x). This is a *secant line*.

PART 4: TOWARDS TANGENTS

1. As you saw in the reading, the ***tangent line is the limit of the secant line as h approaches 0***, which means the two points on the secant line are getting closer and closer. Use the h slider or move the appropriate point to bring the two points closer and closer and see what happens.

As you saw in the reading, the graphical interpretation of the derivative is that the derivative is the slope of the tangent line to the graph of the function. So you've just constructed an approximate derivative calculator! (Approximate because we're using a limiting process on a secant line to approximate a tangent line).

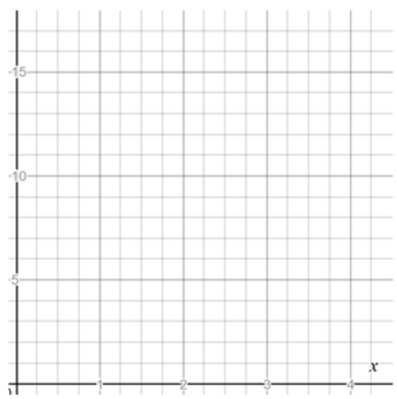
2. Using your tool, determine the derivative at x = 2 (still using f(x) = x^2) by doing the following: set a = 2 and decrease h to a very small number, and see if the slope of this secant line for smaller and smaller h approaches a constant value. Record the slope.

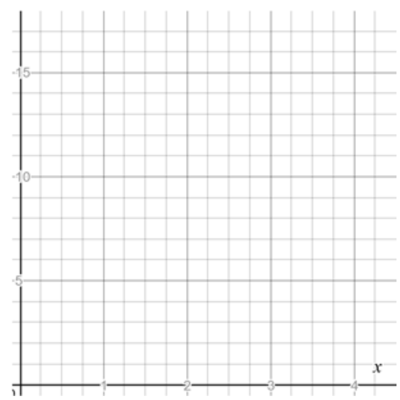
m =

3. Fill out the table below using your Desmos approximate derivative calculator. In this case, **you choose particular x values by setting a** (why)? Make a table of x (change a to various values), the corresponding f(x) at x = a, and the corresponding slope of the limiting secant line (use a very small h). Your book will use the symbol $f'(x)$ to represent the derivative (which is the slope of the tangent line to the graph of $f(x)$ at some point x).

4. On the different sets of axes provided, make a plot of $f(x)$ vs. x (below) and also of $f'(x)$ vs. x, (below, right); label axes.

x	$f(x)$	$f'(x)$
$(= a)$	$(=f(a))$	$(=$ m$)$
0		
1		
2		
3		
4		





5. What do you notice about the shape of the graph of $f'(x)$ vs. x?

6. You can generalize this for any function f(x) just by changing the definition for f(x) in your first line. So instead of f(x) = x^2, you could have f(x) = x^3 - 2x^2 + 3. Set the points far apart, change the function, and see what happens as you bring the points closer together. Just look at the tangent lines as you vary a, and pay attention to how the slope changes. Try again for sin(x), e^x, and other interesting functions.

It is recommended that you save your Demos calculator, since you took the time to build an approximate derivative calculator. To save, you will need a (free) account with Desmos.

PART 5: META-COGNITIVE REFLECTION. See post at program website sites.evergreen.edu for directions/prompts.

PART 6: FURTHER EXTENSION EXPLORATIONS ON LIMITS OF DIFFERENCE QUOTIENTS (if you have time/interest). Particularly recommended if much of the current work is review for you. Ask Krishna for extensions.