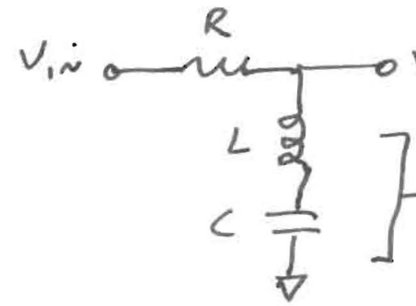
Models of Motion 2015

– Circuits, Week 4 Solutions –

4/21/15

1.26



$\to Z_2 = Z_L + Z_C = \frac{1}{i\omega C} + i\omega L$

Note: this circuit discussed on p 4/4 of the Lab Manual

This circuit is a voltage divider, with the lower (LC) part having $Z_2 = Z_L + Z_C$ = impedance of the capacitor and inductor in series... so

$$V_{out} = \frac{V_{in}\, Z_2}{Z_2 + R}$$ by the voltage divider equation,

or

$$G = \frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_2 + R} \quad \text{with } Z_2 = \frac{1}{i\omega C} + i\omega L$$

It is easiest to re-write Z_2 as a single complex number:

$$Z_2 = \frac{1}{i\omega C} + i\omega L = \frac{-i}{\omega C} + i\omega L = i\left(\omega L - \frac{1}{\omega C}\right)$$

Then

$$\boxed{G = \frac{i\left(\omega L - \frac{1}{\omega C}\right)}{R + i\left(\omega L - \frac{1}{\omega C}\right)}} \qquad \text{Eqn (1)}$$

Arguably this answers the question... but it's better to give $G = |G| e^{i\phi}$, i.e. as a ~~mg~~ magnitude and a phase.

Before we do that, let's see what Eqn (1) tells us:

a) when $\omega L = \frac{1}{\omega C}$, $G = 0$. In other words, the Gain $= 0$ when $\omega^2 = \frac{1}{LC}$ or $\omega = \frac{1}{\sqrt{LC}}$, i.e. $G = 0$ at the resonant frequency $\omega = \frac{1}{\sqrt{LC}}$.

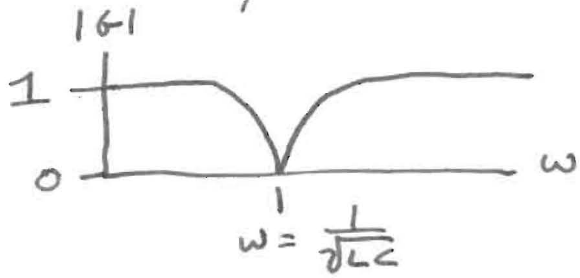
b) As $\omega \to \infty$, $G \to 1$ since $\omega L \gg \frac{1}{\omega C}$, $\omega L \gg R$ when $\omega \to \infty$.

1.26 cont'd

c) As $\omega \to 0$, $G \to 1$, since

$\frac{1}{\omega C} \gg \omega L$ and $\frac{1}{\omega C} \gg R$ when $\omega \to 0$.

So we already can see that $|G|$ behaves like:



Let's write G as a magnitude and a phase:

$G = |G| e^{i\phi}$

Since we already know that $\omega_R = \frac{1}{\sqrt{LC}}$ is the special "resonant" frequency, it will be nicer to write G in terms of $\omega_R = \frac{1}{\sqrt{LC}}$. To do this, let's multiply by:

$$G = 1 \cdot \frac{i(\omega L - \frac{1}{\omega C})}{R + i(\omega L - \frac{1}{\omega C})} = \frac{(\frac{\omega}{L})}{(\frac{\omega}{L})} \frac{i(\omega L - \frac{1}{\omega C})}{R + i(\omega L - \frac{1}{\omega C})}$$

$$= \frac{i(\omega^2 - \frac{1}{LC})}{\omega \frac{R}{L} + i(\omega^2 - \frac{1}{LC})} = \frac{i(\omega^2 - \omega_R^2)}{\omega \frac{R}{L} + i(\omega^2 - \omega_R^2)} = G$$

This makes it easy to get $|G|$:

$$\boxed{|G| = \sqrt{G G^*} = \left[\frac{(\omega^2 - \omega_R^2)^2}{(\omega \frac{R}{L})^2 + (\omega^2 - \omega_R^2)^2} \right]^{\frac{1}{2}}}$$

This is not particularly more illuminating than our earlier expression for G; however, let's look at the phase behavior:

1.26 cont'd

To get ϕ, it would be handy to have G in the form $G = a + ib$, since then $\tan\phi = \frac{b}{a}$.

We can use the usual trick of multiplying G by 1, where $1 = \frac{\text{complex conjugate of denominator}}{\text{complex conj. of denominator}}$:

$$G = \frac{i(\omega^2 - \omega_R^2)}{\omega\frac{R}{L} + i(\omega^2 - \omega_R^2)} \cdot \frac{\omega\frac{R}{L} - i(\omega^2 - \omega_R^2)}{\omega\frac{R}{L} - i(\omega^2 - \omega_R^2)}$$

$$= \frac{(\omega^2 - \omega_R^2)^2 + i\omega\frac{R}{L}(\omega^2 - \omega_R^2)}{(\omega\frac{R}{L})^2 + (\omega^2 - \omega_R^2)^2} = a + ib$$

so $$\frac{b}{a} = \frac{\omega\frac{R}{L}(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_R^2)^2} = \boxed{\frac{\omega R}{L(\omega^2 - \omega_R^2)} = \tan\phi} = \frac{b}{a}$$

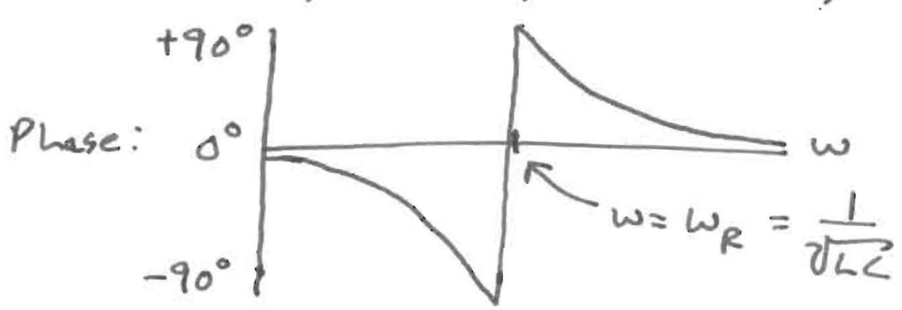
This one is tricky: note that for:

$0 \leq \omega < \omega_R$, $\tan\phi < 0$ and as $\omega \to 0$, $\tan\phi \to 0$
$\omega \to \omega_R$, $\tan\phi \to -\infty$

$\omega_R < \omega \leq \infty$, $\tan\phi > 0$ and as
$\omega \to \infty$, $\tan\phi \to 0$
$\omega \to \omega_R$, $\tan\phi \to \infty$

$\omega = \omega_R$, $\tan\phi$ = undefined.

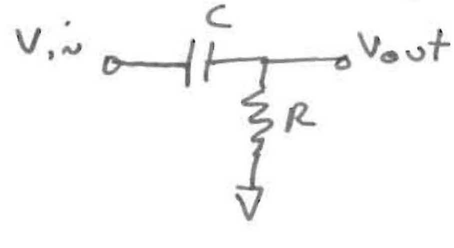
When $\tan\phi < 0$, $\phi < 0°$; when $\tan\phi > 0$, $\phi > 0°$, and when $\tan\phi = \pm\infty$, $\phi = \pm 90°$, so:



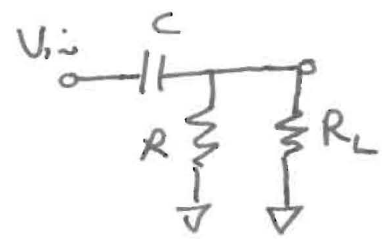
#3, Add. Ex:

Filter must pass frequencies > 20 Hz, and have a 3dB point of $F_{3dB} = 10\ Hz$. Load is $\geq 10k$.

→ if a circuit must pass frequencies above some cutoff, it must be a high-pass filter, which looks like:



or, with the load attached:



For an RC circuit we know $\omega_{3dB} = \frac{1}{RC}$, or

$$F_{3db} = \frac{\omega}{2\pi} = \frac{1}{2\pi RC}.$$

We also know the output impedance of our circuit must be very low compared to R_{Load} so that the circuit behavior doesn't change when we connect the load: $R_{out} \ll R_{Load}$.

Since a voltage source looks like a short circuit,

$$R_{out} = Z_c \parallel R = \frac{R Z_c}{R + Z_c} \ll R_L$$

and from above

$$F_{3dB} = 10\ Hz = \frac{1}{2\pi RC}.$$

Since we need $R_{out} \ll R_L$, we need to know what the ~~smallest~~ largest value of R_{out} is. From above,

$$R_{out} = \frac{R Z_c}{R + Z_c} = \frac{R\left(\frac{1}{i\omega C}\right)}{R + \left(\frac{1}{i\omega C}\right)} \quad \text{(Equation 1)}$$

#3, Add Ex. Cont'd

Looking at Eqn 1, as:

$\omega \to 0$, $R_{out} \to R$ (since $\frac{1}{j\omega C} \to \infty$, and since it's on the top <u>and</u> the bottom, it cancels)

$\omega \to \infty$, $R_{out} \to 0$

There are no special frequencies in between – you could calculate $|R_{out}|$ (really should be $|Z_{out}|$ since it's complex) to see that this is true.

Given that $Z_{out} \leq R$, and we need

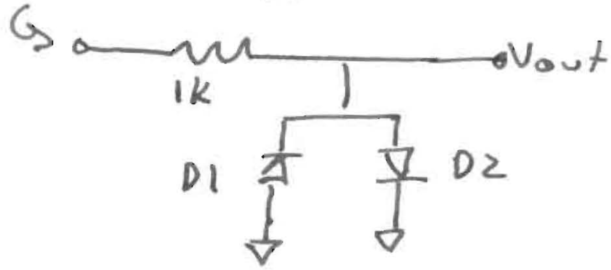
$|Z_{out}| \ll R_L$, i.e. ~~[illegible]~~ $R_L > 10 \cdot |Z_{out}| = 10R$

then $\boxed{R \leq 1K}$ since $R_L = 10K > 10R$.

Since $F_{3db} = 10\,Hz = \frac{1}{2\pi RC}$, $C = \frac{1}{2\pi R \cdot 10\,Hz} \simeq \frac{1}{6}e{-4} = 1.5e{-5}$

or $\boxed{C \simeq 15\mu F} = 15e{-6}\,F$

we're given $V_{in} = 6.3\ V_{rms}$, AC



Diode D2 will conduct when $V_{out} \geq 0.6\ V$ (one diode drop)

so $V_{out} \leq 0.6\ V$.

Diode D1 will conduct (be a short circuit, albeit w/ a 0.6 V drop) when $V_{out} \leq 0.6\ V$.

Hence $\boxed{-0.6\ V \leq V_{out} \leq 0.6\ V}$

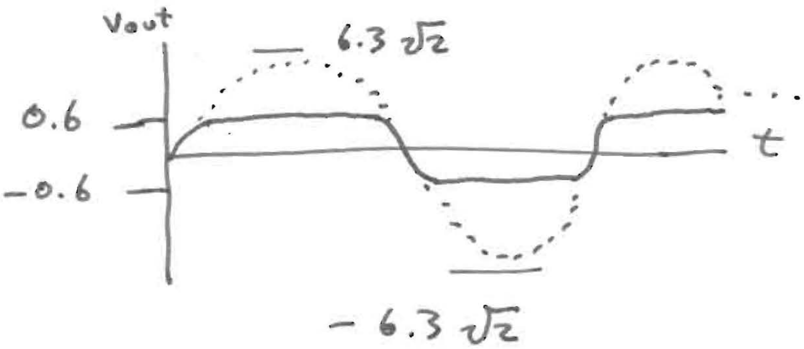
Since $V_{in} = 6.3\ V_{rms}$ (probably from wall voltage, so AC at 60 Hz)

$$V_{in} = \sqrt{2} \cdot 6.3\,V \cdot \cos(2\pi \cdot 60\,Hz \cdot t)$$

since RMS voltage for a sine wave is given by

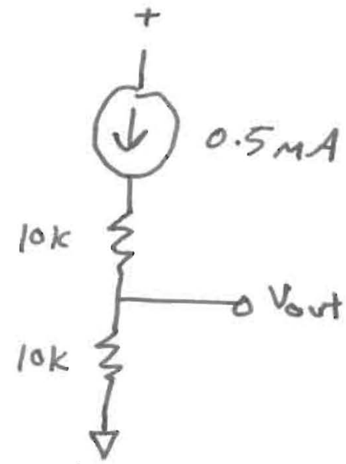
$$\frac{V_{peak}}{\sqrt{2}} = V_{rms}$$

So, roughly, V_{out} will look like



The sinusoidal waveform is clipped at ± 0.6 V.

#2, Add. Ex.



Find the Thevenin equivalent circuit.

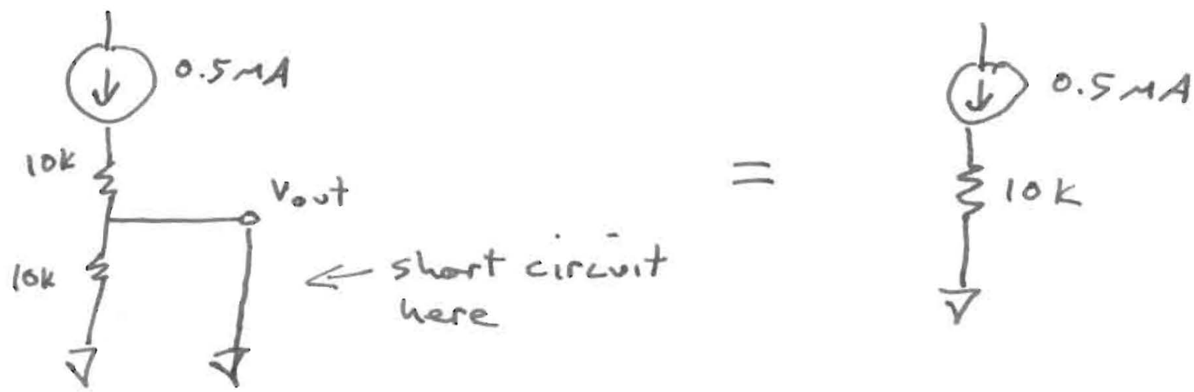
The simplest way to do this is to calculate

$V_{oc} = V_{\text{open circuit}}$ and

$I_{sc} = I_{\text{short circuit}}$.

Then $V_{Th} = V_{oc}$ and $R_{Th} = \dfrac{V_{oc}}{I_{sc}}$.

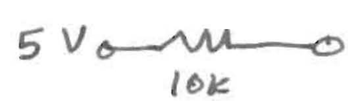
V_{oc} is easy: $V_{oc} = IR = 0.5\,mA \cdot 10k = 5V$



$I_{sc} = 0.5\,mA$, since we're given an ideal current source, so R actually doesn't matter.

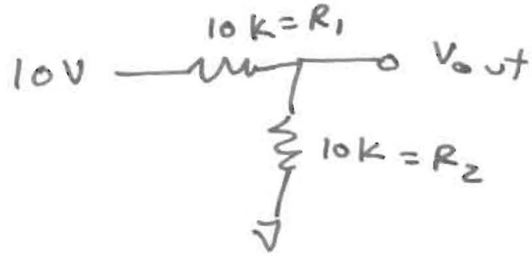
Then $R_{Th} = \dfrac{V_{oc}}{I_{sc}} = \dfrac{5V}{0.5\,mA} = 10k$.

So the equivalent circuit is:



with $\boxed{V_{Th} = 5V, \; R_{Th} = 10k}$

we're asked to compare with the Thevenin equivalent circuit for



Here we can get $V_{oc} = V_{out} = V_{in} \cdot \frac{R_2}{R_1+R_2}$ since the circuit is just a voltage divider.

So $V_{oc} = 10V \cdot \frac{10k}{10k+10k} = 5V$

$I_{sc} = \frac{V_{in}}{R_1} = \frac{10V}{10k} = 1mA$ since when we short the output,

So $R_{TH} = \frac{V_{oc}}{I_{sc}} = \frac{5V}{1mA} = 5K$ so

$\boxed{V_{TH} = 5V,\ R_{TH} = 5K}$ and

for the circuit from #2, which is definitely not equivalent to our result for #3.

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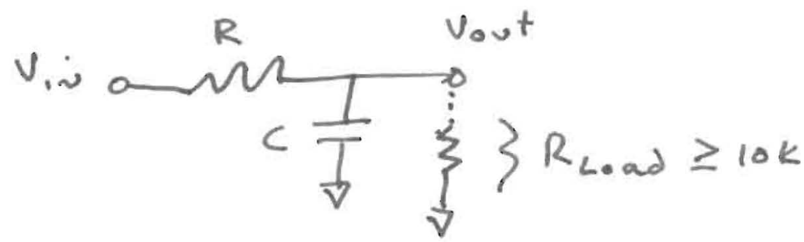
Another way to look at this is that since a voltage source looks like a short (has zero output impedance), $R_{out} = 10k \parallel 10k = 5k$ for this circuit. A current source has ∞ output impedance (current doesn't change no matter what R_{Load} you connect, so $R_{out} \gg$ any R_{Load}, or $R_{out} = \infty$). So for #3, $R_{out} = 10k \parallel \infty \cancel{10k}$ or $R_{out} = 10k$ ... different than #2.

#4 AH. Ex. Scratches are high frequency, so here we want a Low-pass filter. We're given $F_{3db} = \frac{1}{2\pi RC} = 10\text{ kHz}$, $R_{Load} \geq 10\text{K}$, and zero output impedance for the input signal. Hence we have



Similarly to problem #3, we want

$R_{Load} >> |Z_{out}|$, or $R_{Load} \geq 10\,|Z_{out}|$ at all times.

But $Z_{out} = R \parallel Z_c = \frac{R Z_c}{R + Z_c}$, which can never be larger than R.

So $R_{Load} = 10\text{K} \geq 10 \cdot |Z_{out}| \geq 10 \cdot R$ or

$\boxed{R \leq 1\text{K}}$

Then, plugging in to the formula for F_{3db} we get

$$C = \frac{1}{2\pi R F_{3db}} = \frac{1}{2\pi \cdot 1\text{k} \cdot 10\text{kHz}} = \frac{1}{6}e{-7}\text{ F} = 0.15e{-7}\text{ F}$$

or

$$\boxed{\begin{array}{l} C = 15000\text{ pF} = 0.015\,\mu\text{F} \\ R = 1\text{K} \end{array}}$$

and

(capacitances are always given in μf or pf)

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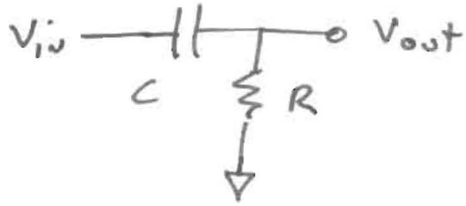
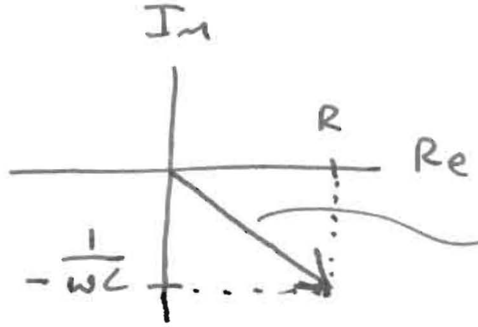


This is a voltage divider, so $G = \frac{V_{out}}{V_{in}} = \frac{R}{R + Z_c}$

where $Z_c = \frac{1}{i\omega C}$ ... so what we really need in order to get $|G|$ is $|R + Z_c|$, since the R in the numerator is already a real number.

$R + Z_c = R + \frac{1}{i\omega C} = R - \frac{i}{\omega C}$. Plotting this gives:



By geometry / Pythagorean theorem, length of this vector will be $\sqrt{R^2 + (\frac{1}{\omega C})^2}$

so $$|G| = \left|\frac{R}{R + Z_c}\right| = \frac{R}{|R + Z_c|} = \boxed{\frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = |G|}$$

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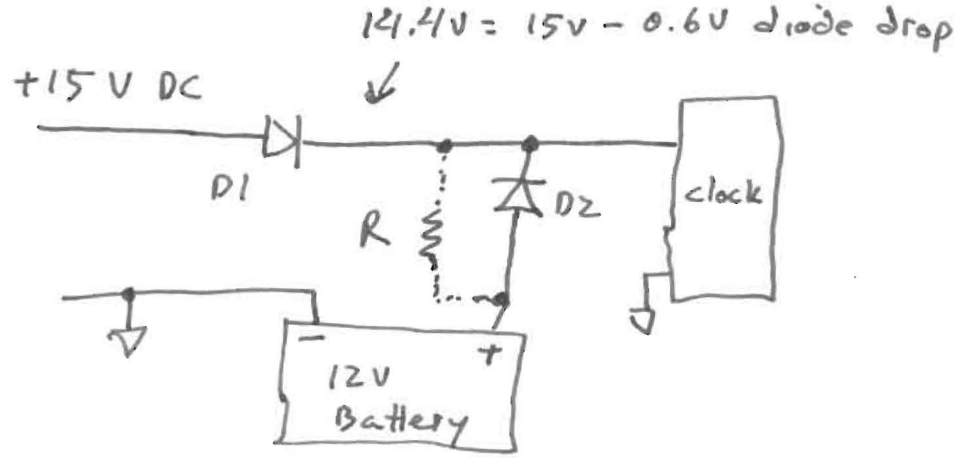

We want the battery to charge at 10 mA when the wall power (15 V dc) is on.

The easiest way to do that would be to add a resistor R as shown. When the wall power is on, and assuming the battery is mostly charged (which is the expectation when you use a "trickle charger" of this sort to maintain charge) then the voltage ΔV across the resistor will be

$$\Delta V = [15\,V - 0.6\,V \text{ diode drop}] - [12\,V \text{ battery}] = 2.4\,V$$

We want $I = 10\,mA$, so $R = \frac{V}{I} = \frac{2.4\,V}{10\,mA} = 240\,\Omega$

so $\boxed{R = 240}$

Note that Diode D1 prevents the battery from discharging into the wall if the power goes off.

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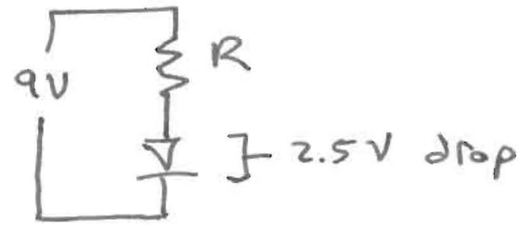
Aside: the problem actually specifies the clock input at $12\,V \le V_{clock} \le 15\,V$ ... but diode D2 will have a diode drop of 0.6 V, so if the wall power goes off then the voltage at the clock will be $\le 11.4\,V$ ...

oops

LED Problem:



→ want

$I = 20\,mA$ max.

A)

ΔV across $R = 9V - 2.5V = 6.5V$

$I = 20\,mA = \frac{\Delta V}{R}$, so $\boxed{R = \frac{\Delta V}{I} = \frac{6.5V}{20\,mA} = 325\,\Omega}$

B) There are lots of ways to figure the power. My way is

$P_{LED} = IV = 20\,mA \cdot 2.5V_{drop} = 50\,mW$

$P_{Resistor} = P_R = I^2 R = (20\,mA)^2 \cdot 325 = 0.13W = 130\,mW$

So $\boxed{\begin{array}{l} P_{LED} = 50\,mW \\ P_R = 130\,mW \end{array}}$

and Fraction of power in LED $= \frac{P_{LED}}{P_{Total}} = \frac{50\,mW}{50+130\,mW} \simeq 0.27$

so $\boxed{\text{efficiency is} \simeq 27\%}$

This is pretty poor efficiency. There are custom chips to do this that provide $\simeq$ 85% efficiency (for a price...)