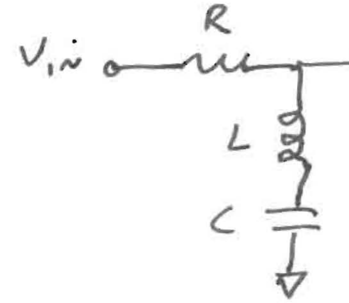
Models of Motion 2015

— Circuits, Week 4 Solutions —

4/21/15

1.26



note: this circuit discussed on p 44 of the Lab Manual

$\rightarrow Z_2 = Z_L + Z_C = \frac{1}{i\omega C} + i\omega L$

This circuit is a voltage divider, with the lower (LC) part having $Z_2 = Z_L + Z_C$ = impedance of the capacitor and inductor in series... so

$$V_{out} = \frac{V_{in}\, Z_2}{Z_2 + R}$$ by the voltage divider equation,

or

$$G = \frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_2 + R} \qquad \text{with } Z_2 = \frac{1}{i\omega C} + i\omega L$$

It is easiest to re-write Z_2 as a single complex number:

$$Z_2 = \frac{1}{i\omega C} + i\omega L = \frac{-i}{\omega C} + i\omega L = i\left(\omega L - \frac{1}{\omega C}\right)$$

Then $$\boxed{G = \frac{i\left(\omega L - \frac{1}{\omega C}\right)}{R + i\left(\omega L - \frac{1}{\omega C}\right)}}$$ Eqn (1)

Arguably this answers the question... but it's better to give $G = |G|e^{i\phi}$, i.e. as a ~~mg~~ magnitude and a phase.

Before we do that, let's see what Eqn (1) tells us:

a) when $\omega L = \frac{1}{\omega C}$, $G = 0$. In other words, the Gain $= 0$ when $\omega^2 = \frac{1}{LC}$ or $\omega = \frac{1}{\sqrt{LC}}$, i.e. $G = 0$ at the resonant frequency $\omega = \frac{1}{\sqrt{LC}}$.

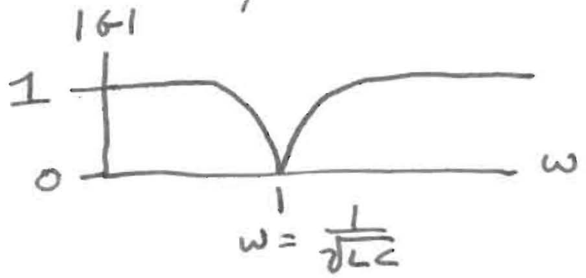
b) As $\omega \to \infty$, $G \to 1$ since $\omega L \gg \frac{1}{\omega C}$, $\omega L \gg R$ when $\omega \to \infty$.

1.26 cont'd

c) As $\omega \to 0$, $G \to 1$, since

$\frac{1}{\omega C} >> \omega L$ and $\frac{1}{\omega C} >> R$ when $\omega \to 0$.

So we already can see that $|G|$ behaves like:



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Let's write G as a magnitude and a phase:

$$G = |G| \, e^{i\phi}$$

Since we already know that $\omega_R = \frac{1}{\sqrt{LC}}$ is the special "resonant" frequency, it will be nicer to write G in terms of $\omega_R = \frac{1}{\sqrt{LC}}$. To do this, let's multiply by:

$$G = 1 \cdot \frac{i(\omega L - \frac{1}{\omega C})}{R + i(\omega L - \frac{1}{\omega C})} = \frac{(\frac{\omega}{L})}{(\frac{\omega}{L})} \frac{i(\omega L - \frac{1}{\omega C})}{R + i(\omega L - \frac{1}{\omega C})}$$

$$= \frac{i(\omega^2 - \frac{1}{LC})}{\omega \frac{R}{L} + i(\omega^2 - \frac{1}{LC})} = \frac{i(\omega^2 - \omega_R^2)}{\omega \frac{R}{L} + i(\omega^2 - \omega_R^2)} = G$$

This makes it easy to get $|G|$:

$$\boxed{|G| = \sqrt{G G^*} = \left[\frac{(\omega^2 - \omega_R^2)^2}{(\omega \frac{R}{L})^2 + (\omega^2 - \omega_R^2)^2} \right]^{\frac{1}{2}}}$$

This is not particularly more illuminating than our earlier expression for G; however, let's look at the phase behavior:

1.26 cont'd

To get ϕ, it would be handy to have G in the form $G = a + ib$, since then $\tan\phi = \frac{b}{a}$.

We can use the usual trick of multiplying G by 1, where $1 = \frac{\text{complex conjugate of denominator}}{\text{complex conj. of denominator}}$:

$$G = \frac{i(\omega^2 - \omega_R^2)}{\omega \frac{R}{L} + i(\omega^2 - \omega_R^2)} \cdot \frac{\omega \frac{R}{L} - i(\omega^2 - \omega_R^2)}{\omega \frac{R}{L} - i(\omega^2 - \omega_R^2)}$$

$$= \frac{(\omega^2 - \omega_R^2)^2 + i\omega \frac{R}{L}(\omega^2 - \omega_R^2)}{(\omega \frac{R}{L})^2 + (\omega^2 - \omega_R^2)^2} = a + ib$$

so $$\frac{b}{a} = \frac{\omega \frac{R}{L}(\omega^2 - \omega_R^2)}{(\omega^2 - \omega_R^2)^2} = \boxed{\frac{\omega R}{L(\omega^2 - \omega_R^2)} = \tan\phi} = \frac{b}{a}$$

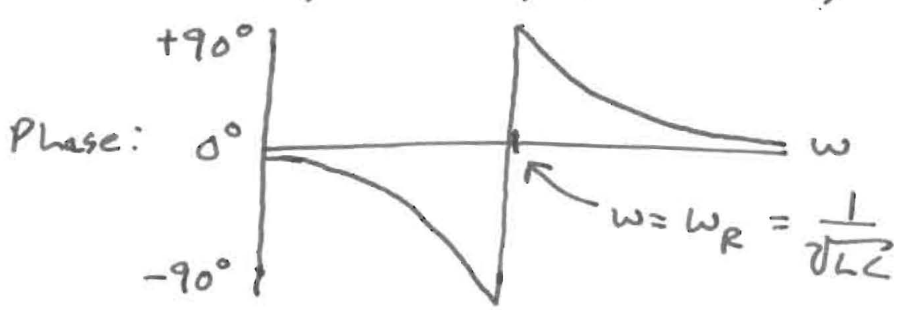
This one is tricky: note that for:

$0 \le \omega < \omega_R$, $\tan\phi < 0$ and as $\omega \to 0$, $\tan\phi \to 0$
$\omega \to \omega_R$, $\tan\phi \to -\infty$

$\omega_R < \omega \le \infty$, $\tan\phi > 0$ and as
$\omega \to \infty$, $\tan\phi \to 0$
$\omega \to \omega_R$, $\tan\phi \to \infty$

$\omega = \omega_R$, $\tan\phi$ = undefined.

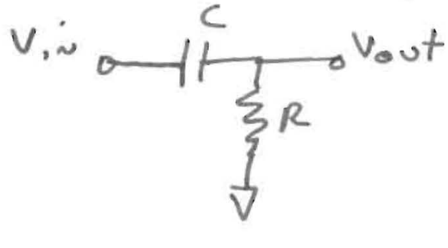
When $\tan\phi < 0$, $\phi < 0°$; when $\tan\phi > 0$, $\phi > 0°$, and when $\tan\phi = \pm\infty$, $\phi = \pm 90°$, so:



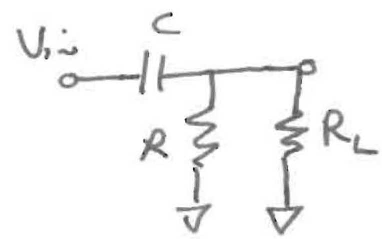
#3, Add. Ex:

Filter must <u>pass</u> frequencies > 20 Hz, and have a 3dB point of $F_{3dB} = 10$ Hz. Load is $\geq$ 10k.

→ if a circuit must pass frequencies above some cutoff, it must be a high-pass filter, which looks like:



or, with the load attached:



For an RC circuit we know $\omega_{3dB} = \frac{1}{RC}$, or

$$F_{3db} = \frac{\omega}{2\pi} = \frac{1}{2\pi RC}.$$

We also know the output impedance of our circuit must be very low compared to R_{Load} so that the circuit behavior doesn't change when we connect the load: $R_{out} << R_{Load}$.

Since a voltage source looks like a short circuit,

$$R_{out} = Z_c \parallel R = \frac{R Z_c}{R + Z_c} << R_L$$

and from above

$$F_{3dB} = 10 \text{ Hz} = \frac{1}{2\pi RC}.$$

Since we need $R_{out} << R_L$, we need to know what the ~~smallest~~ largest value of R_{out} is. From above,

$$R_{out} = \frac{R Z_c}{R + Z_c} = \frac{R\left(\frac{1}{i\omega C}\right)}{R + \left(\frac{1}{i\omega C}\right)} \quad \text{(Equation 1)}$$

#3, Add Ex.
Cont'd   Looking at Eqn 1, as:

$\omega \to 0$, $R_{out} \to R$ (since $\frac{1}{j\omega C} \to \infty$, and since it's on the top <u>and</u> the bottom, it cancels)

$\omega \to \infty$, $R_{out} \to 0$

There are no special frequencies in between - you could calculate $|R_{out}|$ (really should be $|Z_{out}|$ since it's complex) to see that this is true.

Given that $Z_{out} \leq R$, and we need

$|Z_{out}| \ll R_L$, i.e. $R_L > 10 \cdot |Z_{out}| = 10R$

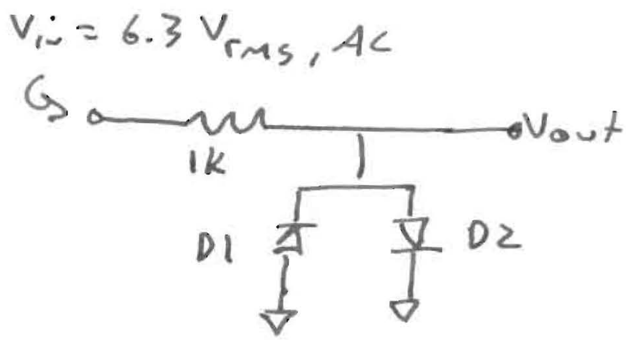
then $\boxed{R \leq 1k}$ since $R_L = 10k > 10R$.

Since $F_{3db} = 10\,Hz = \frac{1}{2\pi RC}$, $C = \frac{1}{2\pi R \cdot 10\,Hz} \simeq \frac{1}{6}e{-4} = 1.5e{-5}$

or $\boxed{C \simeq 15\mu F} = 15e{-6}\,F$

#7, Add. Ex.

we're given



Diode D2 will conduct when $V_{out} \geq 0.6\,V$ (one diode drop)

so $V_{out} \leq 0.6\,V$.

Diode D1 will conduct (be a short circuit, albeit w/ a 0.6V drop) when $V_{out} \leq 0.6\,V$.

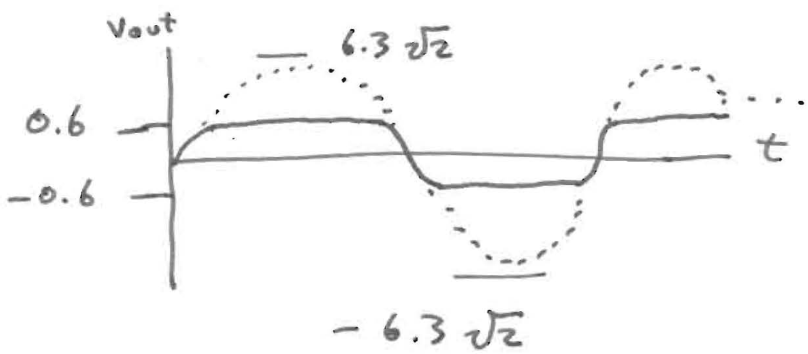
Hence $\boxed{-0.6\,V \leq V_{out} \leq 0.6\,V}$

Since $V_{in} = 6.3\,V_{rms}$ (probably from wall voltage, so AC at 60 Hz)

$$V_{in} = \sqrt{2} \cdot 6.3V \cdot \cos(2\pi \cdot 60Hz \cdot t)$$

since RMS voltage for a sine wave is given by $\frac{V_{peak}}{\sqrt{2}} = V_{rms}$

So, roughly, V_{out} will look like



The sinusoidal waveform is clipped at ± 0.6 V.