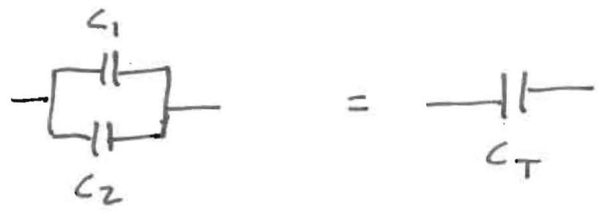
1.16 :



$$Z_{\parallel} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots} \qquad Z_c = \frac{1}{i\omega C}$$

so

$$Z_{\parallel} = \frac{1}{\frac{1}{\left(\frac{1}{i\omega C_1}\right)} + \frac{1}{\left(\frac{1}{i\omega C_2}\right)}} = \frac{1}{i\omega C_1 + i\omega C_2} = \frac{1}{i\omega(C_1 + C_2)}$$

The equivalent impedance is $Z_T = \frac{1}{i\omega C_T}$, so

$$Z_{Total} = Z_{\parallel} = \frac{1}{i\omega C_T} = \frac{1}{i\omega(C_1 + C_2)}$$

($\uparrow$ parallel)

a) Equating $\boxed{C_T = C_1 + C_2}$ gives the rule for adding two capacitors in parallel.

b) Similarly, for

$$Z_{series} = Z_1 + Z_2 = \frac{1}{i\omega C_1} + \frac{1}{i\omega C_2} = \frac{1}{i\omega}\left[\frac{1}{C_1} + \frac{1}{C_2}\right] = \frac{1}{i\omega}\left[\frac{C_1 C_2}{C_1 + C_2}\right]$$

$Z_{Total} = \frac{1}{i\omega C_T}$, and equating the two gives

$$\boxed{C_T = \frac{C_1 C_2}{C_1 + C_2}}$$ for two capacitors in series

Notice that:

$$R_{series} = R_1 + R_2 \qquad C_{series} = \frac{C_1 C_2}{C_1 + C_2}$$

$$R_{\parallel} = \frac{R_1 R_2}{R_1 + R_2} \qquad C_{\parallel} = C_1 + C_2$$

1.17

If $\tilde{A}\tilde{B} = \tilde{C}$, where $\tilde{A}, \tilde{B}, \tilde{C}$ are complex,

then remember that for any complex #, say

$\tilde{A} = a + ib$, we can write it as a magnitude and

a phase, $\tilde{A} = |A| e^{i\phi_A}$ where $|A| = \sqrt{a^2+b^2} = \sqrt{\tilde{A}\tilde{A}^*}$

$$\tan\phi_A = \frac{b}{a}$$

//

Aside: geometrically it may help to visualize this.

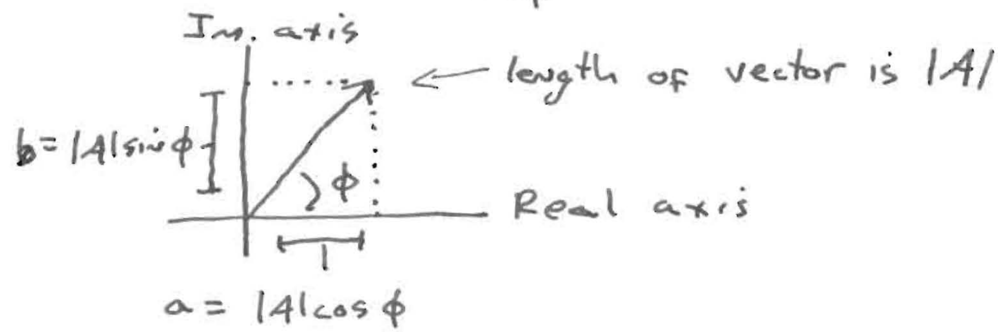
Using Euler's relation, $e^{i\phi} = \cos\phi + i\sin\phi$

$$\tilde{A} = |A| e^{i\phi} = |A|\cos\phi + i|A|\sin\phi$$

$$= \quad a \quad + i \quad b$$

so $a = |A|\cos\phi$
$b = |A|\sin\phi$ and so $\tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{b}{a}$

Visualizing this,



//

Since I can write $\tilde{A} = |A| e^{i\phi_A}$, and the same for $\tilde{B}, \tilde{C}$,

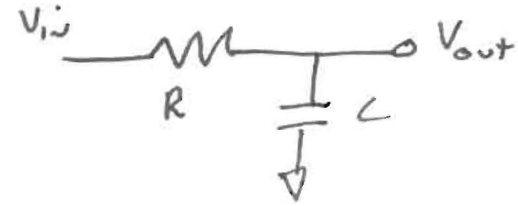
Then

$$\tilde{C} = \tilde{A}\tilde{B} = |A| e^{i\phi_A} \, |B| e^{i\phi_B} = |A|\,|B|\, e^{i(\phi_A+\phi_B)}$$

$$= |C|\, e^{i\phi_C}$$

So for $\tilde{C} = \tilde{A}\tilde{B}$, $|C| = |A|\,|B|$
$\phi_C = \phi_A + \phi_B$

21:



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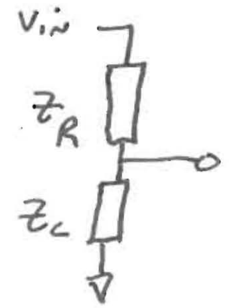


$V_{out} = V_{in} \cdot \frac{Z_C}{Z_R + Z_C}$

The low-pass filter is just a voltage divider with a capacitor for the lower leg. Consequently we can use the voltage-divider equation, and we have

$$V_{out} = V_{in} \cdot \frac{Z_C}{Z_R + Z_C} \quad \text{or} \quad \frac{V_{out}}{V_{in}} = \text{"Gain"} = \frac{Z_C}{Z_R + Z_C}$$

It's called Gain even if it's less than 1, even though then it's an attenuation.

So
$$\tilde{G} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{i\omega RC + 1} = \frac{1}{1 + i\omega RC} \qquad \text{(Eqn 1)}$$

The problem just asks for $|G|$, which is easy:

$|a+ib| = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2+b^2}$ ... all you do is square both the real and imaginary parts. So

$$|G| = \sqrt{G G^*} = \left[\frac{1}{1+i\omega RC} \cdot \frac{1}{1-i\omega RC}\right]^{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{1+(\omega RC)^2}} = |G|}$$

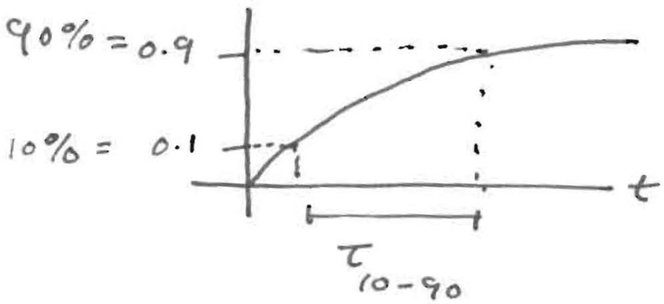
Note that the special frequency where $\omega RC = 1$, and so $|G| = \frac{1}{\sqrt{2}}$ (called the 3db point) is (I guess obviously) at $\omega = \frac{1}{RC}$. Since $\omega = 2\pi f$, this gives $\boxed{F_{3db} = \frac{1}{2\pi RC}}$

The problem doesn't ask for this, but let's do the phase also. Remember that we can write any complex # as

$\tilde{G} = |G| e^{i\phi}$ with $|G| = \sqrt{a^2+b^2}$

$= a + ib$ $\qquad \tan\phi = \frac{b}{a}$. (See problem 1.22)

1.13 10 - 90 % rise time.



$$G = \frac{V_{out}}{V_{in}} = 1 - e^{-t/RC}$$

Curves come in many different shapes, so the 0 - 100 % time can vary a lot. 10% - 90 % is a good (and very standard) rule of thumb that is less sensitive to noise (near 0 or 100 %) or slow approaches to an asymptotic value

For $\tau_{10-90} = (t \text{ when } G = 0.9) - (t \text{ when } G = 0.1)$

we need the times when G reaches different values.

$G = 1 - e^{-t/RC}$ so $e^{-t/RC} = 1 - G$. Taking the ln of both sides gives $\ln e^{-t/RC} = -\frac{t}{RC} = \ln(1-G)$, or $t = -RC \ln(1-G)$.

$$\tau_{10-90} = -RC\left[\ln(1-0.9) - \ln(1-0.1)\right]$$

$$= -RC\left[\ln(0.1) - \ln(0.9)\right] = -RC \ln\frac{0.1}{0.9} = -RC \ln\frac{1}{9}$$

$$= RC \ln 9$$

You can use a calculator to get $\boxed{\tau_{10-90} = 2.2\ RC}$

//

However, I don't. I remember $\ln 10 = 2.3$ (a good # to remember, so you can convert from ln to log)

Using that, $\ln 9 = \ln\left(10 \cdot \left(1 - \frac{1}{10}\right)\right) = \ln 10 + \ln\left(1 - \frac{1}{10}\right)$

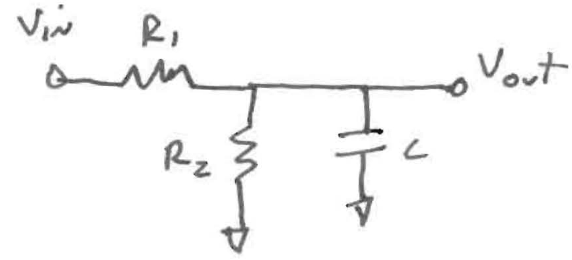
$$= 2.3 + \ln\left(1 - \tfrac{1}{10}\right).$$

Conveniently, the Taylor series for $\ln(1+x) \simeq x + \ldots$ (try it and see), so $\ln\left(1 - \frac{1}{10}\right) \simeq -\frac{1}{10} = -0.1$

So $\ln 9 = 2.3 - 0.1 = 2.2$.

No calculator.

1.14



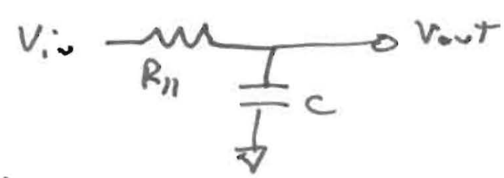
I will do this two ways:

1) If we apply a steady V_{in}, V_{out} will (eventually) also be steady, and for a steady (DC) voltage a capacitor looks like an open circuit (a break in the wire). Another way of saying that is that a steady voltage has zero frequency, and $Z_C = \frac{1}{i\omega C} = \infty$ for $\omega = 0$.

→ if the C is an open circuit, then we're left with a voltage divider of R_1 and R_2, so

$$V_{out}\ (\text{final}) = V_{in} \cdot \frac{R_2}{R_1 + R_2} \qquad (1)$$

The output impedance of that voltage divider is $R_2 \parallel R_1 = \frac{R_1 R_2}{R_1 + R_2} = R_{\parallel}$, so we have a circuit that looks like



and we know that has time dependence

$$V_{out} = V_{in}\left(1 - e^{-t/RC}\right) = V_{in}\left(1 - e^{-t/R_{\parallel}C}\right). \qquad (2)$$

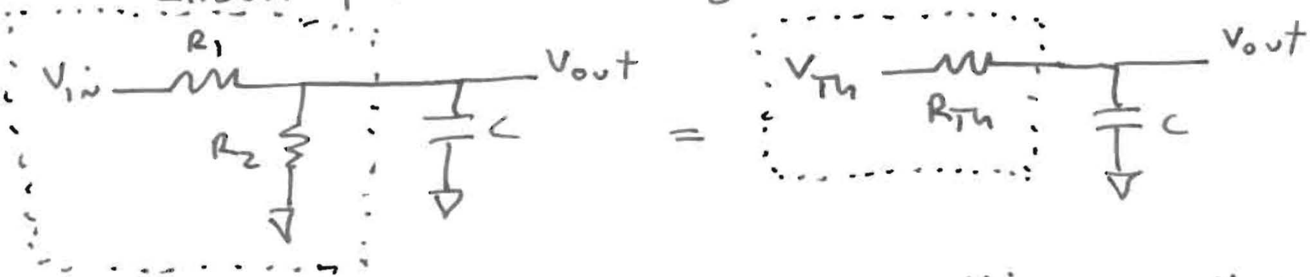
But we just saw that the long-term output voltage is $V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$, so the final time dependence is

$$\boxed{V_{out} = V_{in}\left[\frac{R_2}{R_1 + R_2}\right]\left(1 - e^{-\frac{t}{\left(\frac{R_1 R_2}{R_1 + R_2} C\right)}}\right)}$$

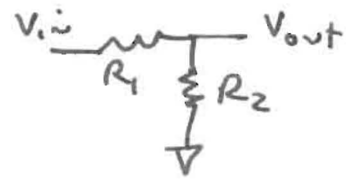
↑ $R_{\parallel} = \frac{R_1 R_2}{R_1 + R_2}$

1.14 cont'd

2) Alternately, just figure out the Thevinin equivalent circuit for the voltage divider:

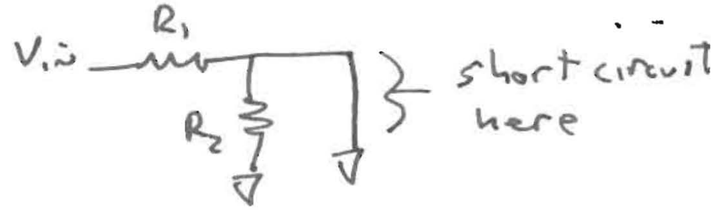


Here the open-circuit voltage for



is $V_{oc} = V_{in} \frac{R_2}{R_1+R_2}$

and the short-circuit current

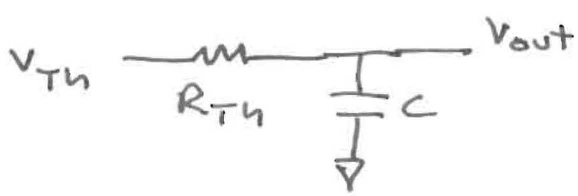


is $I_{sc} = \frac{V_{in}}{R_1}$

Giving $V_{Th} = V_{oc} = V_{in} \frac{R_2}{R_1+R_2}$ } note - this is the same as what I got in part (1)

$R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{R_1 R_2}{R_1+R_2}$ } Ditto - this is $R_{\parallel}$ ...

This leaves us with:



which we know (from the text) has output

$V_{out} = V_{in} (1 - e^{-t/RC})$ where here $V_{in} = V_{Th}$, $R = R_{Th}$

or

$$\boxed{V_{out} = V_{in} \left[\frac{R_2}{R_1+R_2}\right] \left(1 - e^{-\frac{t}{\left(\frac{R_1 R_2}{R_1+R_2}\right) C}}\right)}$$

1.14 cont'd

For $R_1 = R_2 = 10k, \quad R_{11} = 5k$

with $C = 0.1\mu F, \quad R_{11}C = 5K \cdot 0.1\mu F = 5e{-4}\,sec = 0.5\,ms$

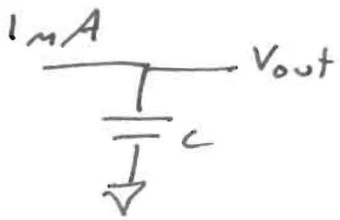
Also, $\frac{R_2}{R_1+R_2} = \frac{1}{2}$. So

$$V_{out} = \frac{V_{in}}{2}\left(1 - e^{-\frac{t}{0.5ms}}\right)$$

note: using the $\tau_{10-90} = 2.2RC$ rule, we know this circuit will be very close to its final voltage within $\simeq 1.1\,ms$.

//

1.15



$Q = CV$ so

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

So $\frac{dV}{dt} = \frac{I}{C} = \frac{1mA}{1\mu F} = \frac{10^{-3}A}{10^{-6}F} = 10^3 \frac{V}{s}$

note: it is never obvious what the units are in electricity and magnetism. If you stick to SI units, it works out, though, and $\frac{A}{F}$ will come out to be $\frac{V}{s}$ ...

This is constant, so

$$\frac{dV}{dt} = \frac{\Delta V}{\Delta t} = \frac{I}{C} = 10^3\ V/s$$

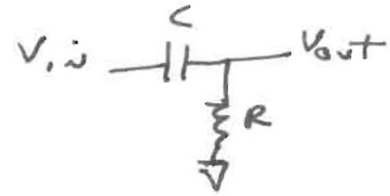
and $\Delta V = \frac{I}{C}\Delta t = 10^3\ V/s \cdot \Delta t$

Solving for $\Delta V = 10$ Volts gives

$$\Delta t = \frac{C}{I}\Delta V = 0.01\,s = 10\,ms$$

1.22 Show that for low- and high-pass filters, ϕ is within $6°$ of its asymptotic value at $f = 0.1\, f_{3db}$ and $f = 10\, f_{3db}$.

1) Let's do the high-pass filter first:



It's a voltage divider geometry, so we know $V_{out} = V_{in}\left(\frac{Z_R}{Z_R + Z_C}\right)$ or $\tilde{G} = \frac{Z_R}{Z_R + Z_C}$, so

$\tilde{G} = \frac{R}{R + \frac{1}{i\omega C}} = |G|\, e^{i\phi}$, since we can write any complex # as a magnitude ~~of~~ time a phase factor.

For this problem we only need the phase, but let's get the magnitude anyway, since it's easy:

$$|G| = \sqrt{\tilde{G}\tilde{G}^*} = \sqrt{\frac{R^2}{R^2 + \frac{1}{(\omega C)^2}}} = \sqrt{\frac{1}{1 + \frac{1}{(\omega RC)^2}}} = \sqrt{\frac{(\omega RC)^2}{1 + (\omega RC)^2}}$$

$$|G| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$ (high pass)

note:

for	$\lvert G\rvert$	
$\omega = 0$	0	← blocks low freqs
$\omega RC = 1$	$\frac{1}{\sqrt{2}}$	← 3db point
$\omega = \infty$	1	← passes high freqs

We can write

$$\tilde{G} = a + ib = |G|\, e^{i\phi} = |G| \cos\phi + i\, |G| \sin\phi$$

so $a = |G| \cos\phi$, $b = |G| \sin\phi$, and $\tan\phi = \frac{b}{a}$.

Hence to get ϕ, we need to write $\tilde{G}$ in a way that makes it easy to get a and b.... but $\tilde{G} = \frac{R}{R + \frac{1}{i\omega RC}}$ has complex #s on the bottom, which is inconvenient. We could make the bottom (denominator) real by multiplying it by it's complex conjugate, and that's fine to do if we multiply the top (numerator) by the same thing:

1.22 Cont'd

$$\tilde{G} = \frac{R}{R + \frac{1}{i\omega C}} \cdot 1 = \frac{R}{R + \frac{1}{i\omega C}} \cdot \frac{R - \frac{1}{i\omega C}}{R - \frac{1}{i\omega C}} = \frac{R^2 - \frac{R}{i\omega C}}{R^2 + \frac{1}{(\omega C)^2}}$$

$$= \frac{1 - \frac{1}{i\omega RC}}{1 + \frac{1}{(\omega RC)^2}} = \frac{1}{\left(1 + \frac{1}{(\omega RC)^2}\right)} + i \frac{\frac{1}{\omega RC}}{1 + \frac{1}{(\omega RC)^2}}$$

$$= \quad a \quad + i \quad b$$

Since $\tilde{G} = a + ib = |G| e^{i\phi}$ with $\tan\phi = \frac{b}{a}$, we have

$$\boxed{\tan\phi = \frac{1}{\omega RC}}$$

(for high-pass)

note that $\tan\phi = 0$ when $\phi = 0$
$= 1$ when $\phi = 45°$
$= \infty$ when $\phi = 90°$

since → $\tan\phi = \frac{\sin\phi}{\cos\phi}$

Consequently we can complete the previous table:

High pass filter:

	ω	$\lvert G\rvert$	ϕ
	0	0	90°
where $\omega RC = 1$ →	ω_{3db}	$\frac{1}{\sqrt{2}}$	45°
	∞	1	0°

The problem asks about the phase at $f = 0.1 f_{3db}$ and $10 f_{3db}$

This is the same as asking about the phase at

$\omega = 0.1\, \omega_{3db}$ and $10\, \omega_{3db}$ $= \frac{1}{10RC}$ and $\frac{10}{RC}$

Since $\omega_{3db} = \frac{1}{RC}$ (remember, $\omega RC = 1$ for ω_{3db}).

So we want to know $\tan\phi = 10$ for $0.1\, \omega_{3db}$

$= \frac{1}{10}$ for $10\, \omega_{3db}$

You can solve this with your calculator: for		
$0.1\,\omega_{3db}$	$\tan\phi = 10$	$\Rightarrow \phi = 84°$
$10\,\omega_{3db}$	$\tan\phi = \frac{1}{10}$	$\Rightarrow \phi = 6°$

// aside: solving $\tan\phi = 10$ by hand is tougher, but for $\tan\phi = \frac{1}{10} = 0.1$ it's easy: $\tan\phi = \frac{\sin\phi}{\cos\phi}$.

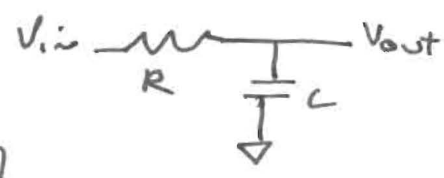
$\cos(0.1) \simeq 1$, $\sin(0.1) \simeq 0.1$, so $\tan(0.1) \simeq 0.1 \ldots$

so $\phi \simeq 0.1$ radians. There are $360°$ in 2π radians, and $2\pi \simeq 6$, so 1 radian $\simeq 60°$ and 0.1 rad $\simeq 6°$.

note: the Taylor series for $\tan x = x + \cdots$, as may make sense to you given that $\sin x \simeq x - \cdots$.

//

2) For the low-pass filter



it's the same: $V_{out} = V_{in}\left[\frac{Z_C}{Z_R + Z_C}\right]$

or

$$\tilde{G} = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{1 + i\omega RC}$$

we can immediately get the magnitude:

$\tilde{G} = |G| e^{i\phi}$, and $|G| = \sqrt{\tilde{G}\tilde{G}^*} = \sqrt{\frac{1}{1+i\omega RC} \cdot \frac{1}{1-i\omega RC}}$

$$= \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

which has, for

	$\|G\|$	
$\omega = 0$	1	← passes low frequencies
$\omega RC = 1$	$\frac{1}{\sqrt{2}}$	← 3db point
$\omega = \infty$	0	← blocks high frequencies

1.22 Con't'd

We can play the same trick as before to get the phase:

$$\tilde{G} = \frac{1}{1+i\omega RC} \cdot 1 = \frac{1}{1+i\omega RC} \cdot \frac{1-i\omega RC}{1-i\omega RC} = \frac{1-i\omega RC}{1^2+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + i\,\frac{(-\omega RC)}{1+(\omega RC)^2}$$

$$= a + i\,b$$

so $\boxed{\tan\phi = \frac{b}{a} = -\omega RC}$ Again, $\tan(0) = 0$

$\tan(-90°) = -\infty$, so:

$\tan(-45°) = -1$

we can complete the table:

Low-pass filter:

	ω	$\lvert G \rvert$	ϕ
	0	1	0
where $\omega RC = 1$ →	ω_{3db}	$\frac{1}{\sqrt{2}}$	$-45°$
	∞	0	$-90°$

note phase shifts are reversed and opposite in sign to high-pass filter

For $\omega = 0.1\,\omega_{3db} = 0.1 \cdot \frac{1}{RC}$,

$\tan\phi \simeq -\frac{0.1}{RC} \cdot RC = -0.1$, and $\tan\phi \simeq \phi + \cdots$ so

$\phi \simeq -0.1$ radian, $60° = 1$ radian, so $\phi_{0.1\,3db} = -6°$

Then, using a calculator we can get

$\tan\phi = -(10\,\omega_{3db})RC = -10 \Rightarrow \phi = -84°$ for $10\,\omega_{3db}$

$\tan\phi = -(0.1\,\omega_{3db})RC = -0.1 \Rightarrow \phi = -6°$ for $0.1\,\omega_{3db}$

Note that for both high and low-pass filters, the large phase shift occurs when the attenuation is high.

1.24 From problem 1.22 we have, for a low-pass filter,

$$|G| = \frac{1}{\sqrt{1+(\omega RC)^2}} .$$

We're asked for the ω where $|G| = \frac{1}{2}$ (i.e. $\frac{V_{out}}{V_{in}} = \frac{1}{2}$, the 6db point). Solving,

$$|G| = \frac{1}{2} = \frac{1}{\sqrt{1+(\omega RC)^2}} \implies 1+(\omega RC)^2 = 4 \implies \omega RC = \sqrt{3}$$

So the 6db point will be at

$$\boxed{f_{6db} = \frac{\omega}{2\pi} = \frac{\sqrt{3}}{2\pi RC} = \sqrt{3}\, \omega_{3db}}$$

We also have, from problem 1.22, that

$\tan\phi = -\omega RC$, so for $\omega = \frac{\sqrt{3}}{RC}$

$$\tan\phi = -\frac{\sqrt{3}}{RC} RC = -\sqrt{3}$$

For this you need a calculator:

$$\boxed{\tan\phi = -\sqrt{3} \implies \phi_{6db} = -60^\circ}$$

↗

how embarrassing - I could have done it by hand:

of course $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, so $\tan 60^\circ = \sqrt{3}$...

it's a good idea to always remember the sin and cos of 0°, 30° $(\frac{\pi}{6})$, 45° $(\frac{\pi}{4})$, 60° $(\frac{\pi}{3})$, and 90° $(\frac{\pi}{2})$.