Models of Motion 14/15
Circuits Midterm

Solutions

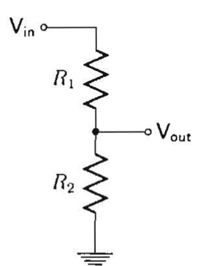
Your name: ______________________________
Last, First

Instructions:

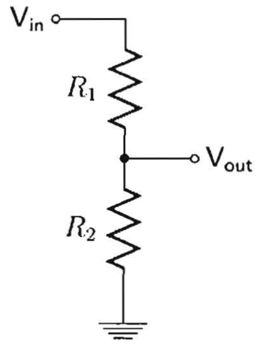
1. **CLOSED BOOK.**
2. **HONOR CODE**: you must do your own work, and use **no** external resources (notes, etc.)
3. **READ** the whole test first. 100 points total.
4. **STRATEGY**:
   a. Do the easy problems first! Get all the points you can!
   b. Do the hard problems next; if really stuck, circle it to come back to it later, then move on to the next question. Don't just work away all your time on one problem.
5. **SHOW** your work. An answer with no work = zero points.

Basics:

1. Fill in the following formulas (15 points):

<u>Two</u> formulas for the power dissipated in a resistor: First formula:	$P = I^2 R$ Both obtained from $P = IV$ and $V = IR$
<u>Two</u> formulas for the power dissipated in a resistor: Second formula:	$P = \frac{V^2}{R}$
Formula for the total resistance of two resistors in parallel:	$R_{parallel} = \frac{R_1 R_2}{R_1 + R_2}$
Formula for V_{out} from a voltage divider 	$V_{out} = V_{in} \cdot \frac{R_2}{R_1 + R_2}$
Impedance of an Inductor, Z_L	$Z_L = i\omega L$ obtained using $V = L\frac{dI}{dt}$ and choosing $I = I_0 e^{i\omega t}$
Impedance of a Capacitor, Z_C	$Z_C = \frac{1}{i\omega C}$ obtained using $I = C\frac{dV}{dt}$ and choosing $V = V_0 e^{i\omega t}$
Ohm's Law	$V = IR$

2. (10 pts) Use Ohm's Law to prove the voltage divider equation for V_{out} given a known V_{in}.



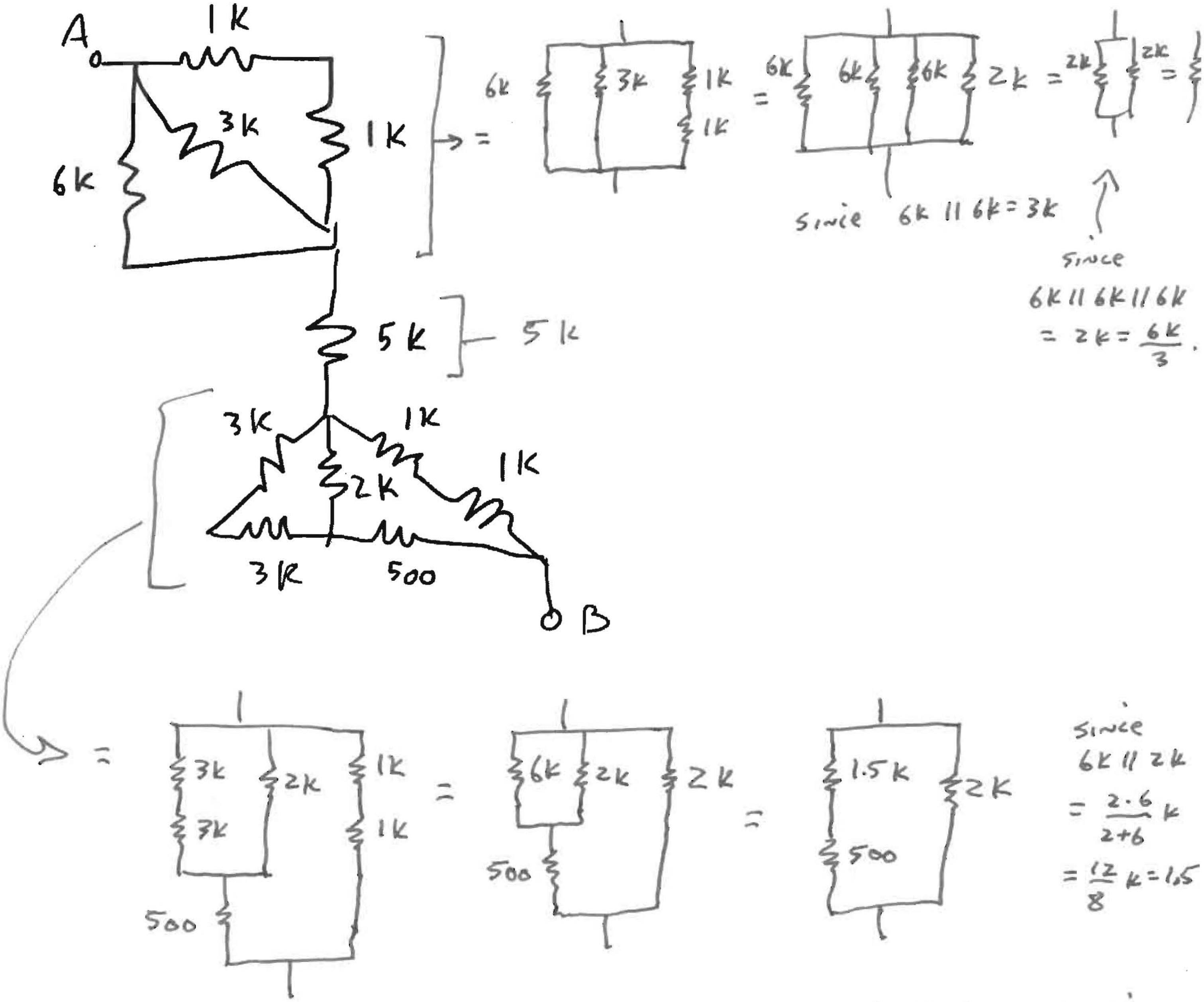
$V_{out} = I R_2$

$$I = \frac{\Delta V}{R_{total}} = \frac{(V_{in} - 0)}{(R_1 + R_2)} = \frac{V_{in}}{R_1 + R_2}$$

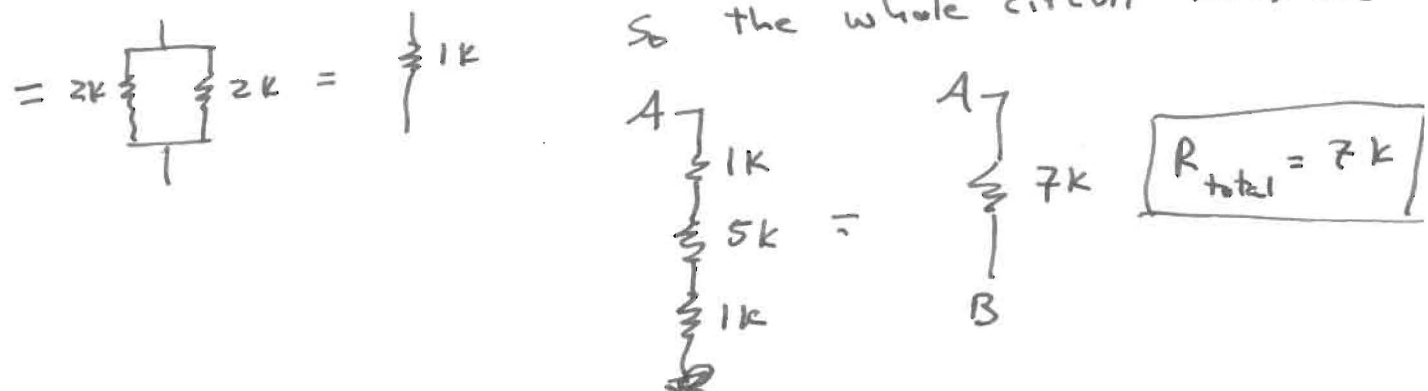
↗ series resistance of R_1 and R_2

$$\boxed{V_{out} = V_{in} \cdot \frac{R_2}{R_1 + R_2}}$$

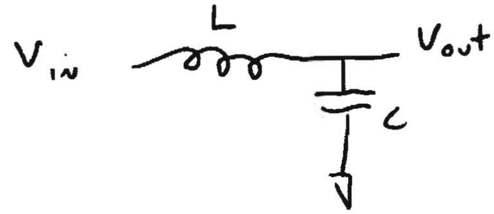
3. (15 points total) Find the equivalent resistance between points A and B of the following circuit. Show enough of your work that I can see you knew what you were doing.



So the whole circuit looks like



4. (15 pts) What is the magnitude of the gain, $|G| = \left|\frac{V_{out}}{V_{in}}\right|$ for the following circuit? Include a plot of |G| vs the frequency ω.



It's a voltage divider geometry, so

$$V_{out} = V_{in} \cdot \frac{Z_C}{Z_L + Z_C} \quad \text{or} \quad G = \frac{Z_C}{Z_L + Z_C} = \frac{\frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}}$$

Multiplying by $1 = \frac{i\omega C}{i\omega C}$ gives

$$G = \frac{i\omega C}{i\omega C} \cdot \frac{\frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}} = \frac{1}{-\omega^2 LC + 1} = \frac{1}{1 - \omega^2 LC} = \frac{1}{1 - \left(\frac{\omega}{\omega_R}\right)^2}$$

where $\omega_R = \frac{1}{\sqrt{LC}}$

So G is not complex, and its magnitude is

$$|G| = \left|\frac{1}{1 - \left(\frac{\omega}{\omega_R}\right)^2}\right| \quad \text{with } \omega_R = \frac{1}{\sqrt{LC}}$$

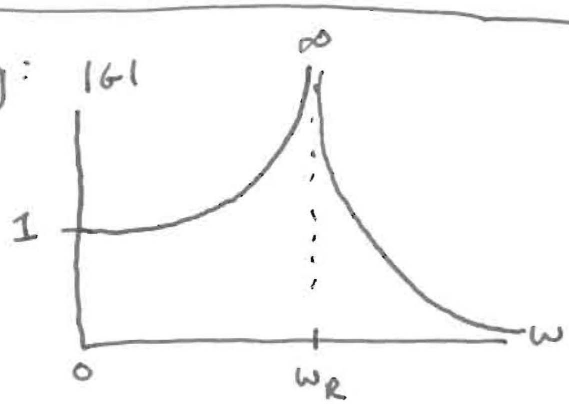
Note: as

$\omega \to 0$, $|G| \to 1$

$\omega \to \omega_R$ $|G| \to \infty$

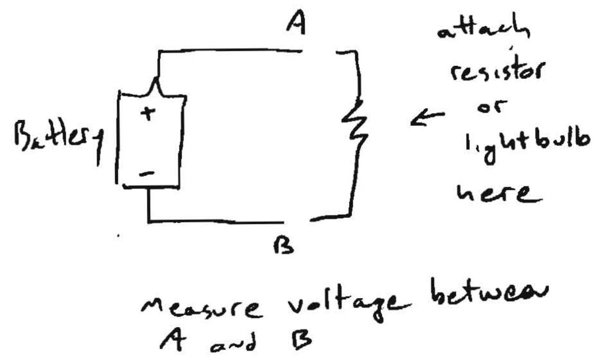
$\omega \gg \omega_R$ $|G| \simeq \frac{\omega_R^2}{\omega^2}$

Plotting:



So this circuit is a sort of low-pass filter, with a resonance at $\omega = \omega_R$.

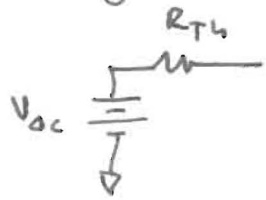
5. (10 pts) Real batteries are not ideal voltage sources. You have a flashlight with a battery that puts out 3.0V. Your battery is pretty old though; when you measure, the voltage from the flashlight it is still 3.0V, but when you connect a 10Ω resistor and measure again the voltage it is only 2.5V.



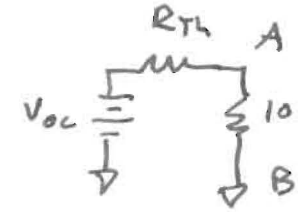
a. What is the Thevenin equivalent circuit for the flashlight battery? For a battery, the Thevenin resistance is known as the "internal resistance".

1) when we measure V, we get 3.0 V, so $V_{oc} = V_{Th} = 3.0$ V

2) Drawing the battery as a Thevenin equivalent gives



connecting a 10 Ω resistor gives



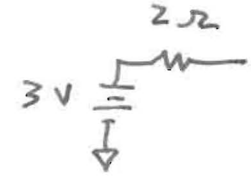
which is a voltage divider, with $V_A = \frac{10}{10 + R_{Th}} \cdot V_{oc}$

or $2.5\text{ V} = 3.0\text{V} \cdot \frac{10}{10 + R_{Th}}$; rearranging gives $\frac{6}{5} = 1 + \frac{R_{Th}}{10}$

or $R_{Th} = 2\ \Omega$. So

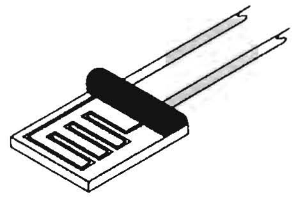
$V_{Th} = 3.0$ V $R_{Th} = 2\ \Omega$

and the Thevenin equivalent for the battery is



Note: 2 Ω is a lot - a flashlight bulb only has a resistance of ≃ 7 Ω, so an old batter results in a signifigant power reduction. (by almost half - $\frac{49}{81}$ ... can you explain why?)

6. (15 pts) Unburned fuel in engine exhaust is a major form of air pollution. In modern cars a sensor measures the exhaust gas and then the fuel/air ratio going into the engine is adjusted so that there is always extra oxygen in the exhaust (allowing for complete combustion). The ceramic sensor element must be hot (~ 650 °C) in order to operate properly, and of course it is cold when the car starts… so there is a heater, which gets it hot quickly so the engine management can work; heating also prevents condensation from ruining the sensor by keeping it hotter than the exhaust.



You are designing such a sensor. You heat it with a resistive strip on the ceramic heater element, as shown. You can power the heater from the 12V car battery.

a. The heater must put out 25W of heat when it's cold in order to get the sensor up to temperature quickly.

   i. What resistance should you make the thin-film platinum strip?

$P = \frac{V^2}{R}$, so $25W = \frac{(12V)^2}{R} \Rightarrow R = \frac{144}{25} \approx \frac{150}{25} = 6\,\Omega$

So I'd make $R = 6\,\Omega$ (or $6 - \frac{6}{25} \approx 6 - \frac{5}{25} = 5.8$ if you really care)

   ii. What current must the strip be capable of handling without breaking?

$P = IV$ so $I = \frac{25W}{12V} = 2\frac{1}{12}A \approx 2.08A$

or you could use $V = IR$ so

$I = \frac{V}{R} = \frac{12V}{6\,\Omega} = 2A$

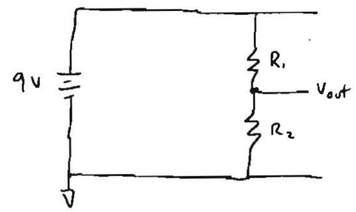
b. Your colleague tells you it will take 6W of power to keep water from condensing on the sensor once it's up to temperature. The resistance of Pt goes up by 3.4x between 0°C and 650°C; will you have sufficient power, or will you need an extra circuit to solve this problem?

$P = \frac{V^2}{R}$ so if R goes to $3.4R_0$, $P = \frac{V^2}{3.4R_0} = \frac{1}{3.4} \cdot \frac{V^2}{R_0} = \frac{1}{3.4} P_0$

$= \frac{1}{3.4} 25W \approx \frac{1}{3} \cdot 24W = 8W$

which is $\geq 6W$, so you're fine.

7. (20 pts) Biasing. An audio output signal is sinusoidal and symmetric about V = 0. Say you want to amplify this using a device powered by a 9V battery… batteries have only two terminals, so if you use one for ground, then you would only be able to amplify half your signal (say the half for V > 0) and would lose the other half.

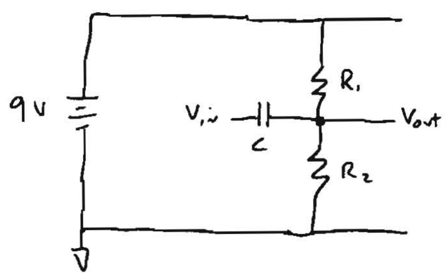


a. Choose resistors that give a V_{out} of 5.0V with a total current of 10mA. (if you care about such things, the 5.0V instead of the 4.5V midpoint is to accommodate a diode drop in a transistor we'd hook up later; the need for 10mA is also to accommodate the transistor).

1) Total current $I = \frac{V}{R_{tot}} = \frac{9V}{R_1+R_2} = 10mA$, so $R_1+R_2 = \frac{9V}{0.01A} = 900\,\Omega$

2) $\frac{V_{out}}{V_{in}} = \frac{5V}{9V} = \frac{5}{9} = \frac{R_2}{R_1+R_2} = \frac{R_2}{900} \Rightarrow$ $R_2 = 500\,\Omega$, $R_1 = 400\,\Omega$ and so, since $R_1+R_2 = 900$

To couple in your audio signal, we can use a capacitor:



it's a high-pass filter geometry, since at $\omega=0$ $Z_C=\infty$ and the capacitor will block V_{in}.

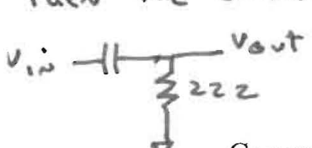
b. For a constant $V_{in} = 0$, what is V_{out}? Is this a high-pass or a low-pass filter?

For constant V_{in}, $\omega=0$ and $Z_C = \frac{1}{i\omega C} = \infty$, ie. the capacitor is an open circuit. So we can ignore it and are left with the voltage divider, which puts out $V_{out} = 5.0\,V$

c. Assuming V_{in} can vary, choose a capacitor value that gives you a 3dB point of about 20Hz (the usual for audio applications). Hint: what is the input impedance of the resistors?

The input impedance of the voltage divider looks like $R_1 \parallel R_2$, since the voltage source will look like a short circuit. $R_{tot} = R_1 \parallel R_2 = \frac{400 \cdot 500}{900} = \frac{2000}{9} = 222\,\Omega$

Then the circuit can be re-drawn as

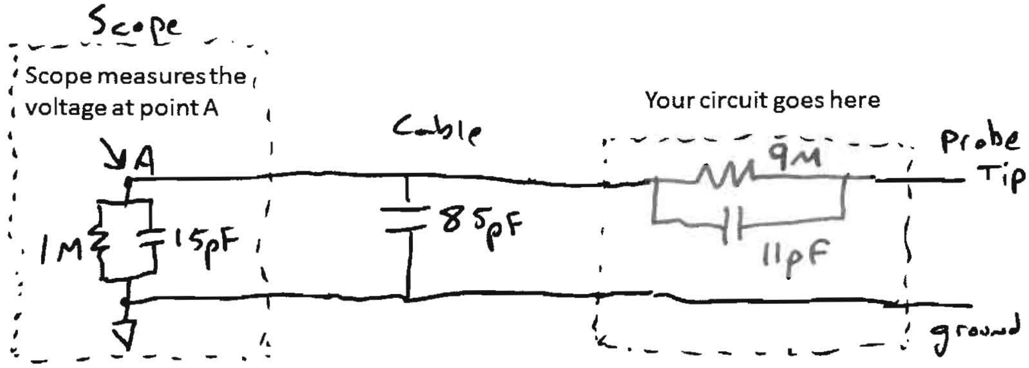


which is a high pass filter with $F_{3db} = \frac{1}{2\pi RC} \Rightarrow C = \frac{1}{2\pi R F_{3db}} = \frac{1}{2\pi \cdot 222 \cdot 20} = \frac{1}{6.3 \cdot 4.4 \cdot 10^3} \simeq \frac{1}{2.8e4} \approx 0.36e-4$

or $C \simeq 36\,\mu F$

Congratulations – you now have your audio signal conveniently centered at 5.0V, so you can amplify both the whole signal.

8. (Extra Credit: 20 points) Oscilloscope 10X Probe. An oscilloscope has in input impedance of 1M in parallel with 15pF. In addition, the shielded cable you use has substantial capacitance, about 30pF/foot – say 85pF for the ~ 3ft cable. Often you want to look at circuits where attaching 1M or a significant capacitance is a problem (it can make circuits oscillate, among other things). We want to design a probe for the oscilloscope that has a circuit in the end that helps fix this.

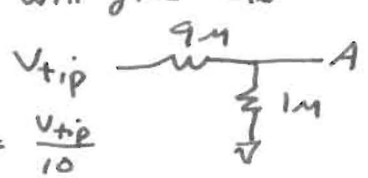


a. Add a circuit component (give the type and value) in the probe tip area that makes the DC ($\omega = 0$) input impedance 10M for the combination of cable and oscilloscope. What is the price you pay for this in terms of the voltage reading on the oscilloscope (measured at point A)? Show any equations you use.

The input impedance (looking into the probe tip) will be

$Z_{tot} = Z_{probe} + Z_{scope+cable}$. at $w=0$, the capacitors are open circuits,

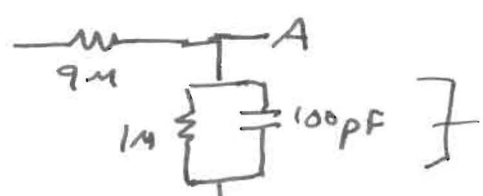
so $Z_{tot} = R_{probe} + R_{scope+cable} = R_{probe} + 1M$. So adding a series resistor $R_{probe} = 9M$ will give $Z_{in} = 10M$ at $w=0$. This will make a voltage divider



$V_A = V_{tip} \cdot \frac{1M}{1M+9M} = \frac{V_{tip}}{10}$, so the scope will only measure $\frac{1}{10}$ of the voltage at the probe tip.

b. At high frequency your new component will cause trouble in conjunction with the regular input impedance of the oscilloscope and cable. What sort of trouble, and at approximately what frequency will it start to become bad? Do not do complicated calculations – make a simple argument.

1) the input capacitance is $15pF \parallel 85pF = 100pf$, so the circuit looks like



when $Z_c \lessapprox 1M$, the lower leg of this voltage divider will start to get small, reducing the voltage you measure

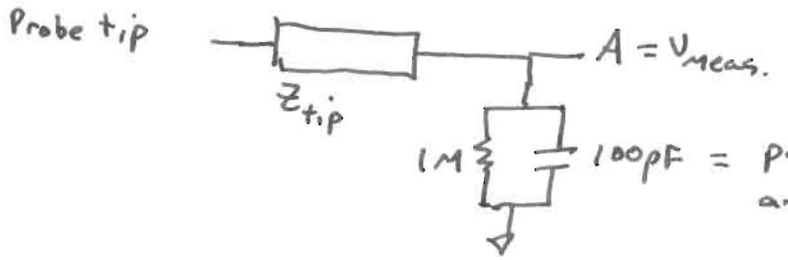
$|Z_c| = \left|\frac{1}{iwc}\right| \simeq 1M$

when $w \simeq \frac{1}{1M \cdot 100pF} = 10^4$

$f = \frac{10^4}{2\pi} \simeq 1.5\ kHz$ which is really low! (ie., a big problem)

c. Add a component (give the value and show the position) to your circuit in the probe tip that results in the oscilloscope measuring the same voltage ratio at all frequencies ω. Show he equation(s) you use to make your choice.

The voltage ratio $\frac{V_{meas\ at\ A}}{V_{tip}} = G = \frac{1}{10}$ at $\omega = 0$; we now want that at all frequencies. We have:



= parallel capacitance of cable and 'Scope

So we need

$$G = \frac{1}{10} = \frac{1M \parallel 100pF}{Z_{tip} + (1M \parallel 100pF)} = \frac{Z_{scope+cable}}{Z_{tip} + Z_{scope+cable}}$$

or

$$10 = 1 + \frac{Z_{tip}}{Z_{scope+cable}} \Rightarrow Z_{tip} = 9 \cdot Z_{scope+cable}$$

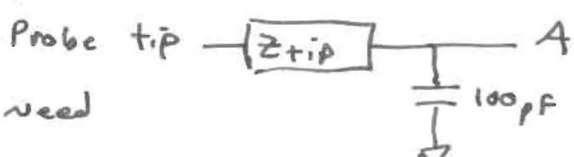
or

$$Z_{tip} = 9\left[1M \parallel 100pF\right] = 9\left[1M \parallel \frac{1}{i\omega 100pF}\right] = \left[9M \parallel \frac{9}{i\omega 100pF}\right]$$

$$= \left[9M \parallel \frac{1}{i\omega(\frac{100}{9})pF}\right] = \left[9M \parallel \frac{1}{i\omega 11pF}\right] = 9M \parallel (11pF)$$

So we need to add a capacitor in the tip in parallel with the 9M resistor, with value $C \simeq 11pF$. In practice, the length of cable may vary, and hence the input capacitance, so the capacitor in the tip is adjustable.

Note: an easy way to think about all this is that at high frequencies the $Z_c = \frac{1}{i\omega \cdot 100pF}$ looks very small, and we can ignore the 1M resistor. Then we have



and to keep $G = \frac{1}{10}$ we'll need

$Z_{tip} \simeq 9 \cdot Z_{100pF}$, or $C_{tip} = \frac{100pF}{9}$.