Calculus, Second Edition
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CHAPTER 0

Highlights of Calculus

0.1 Distance and Speed // Height and Slope

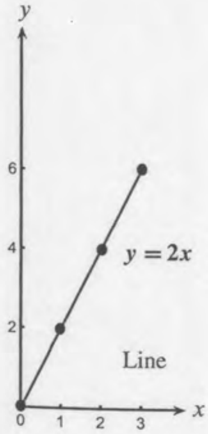
Calculus is about functions. I use that word "functions" in the first sentence, because we can't go forward without it. Like all other words, we learn this one in two different ways: We *define* the word and we *use* the word.

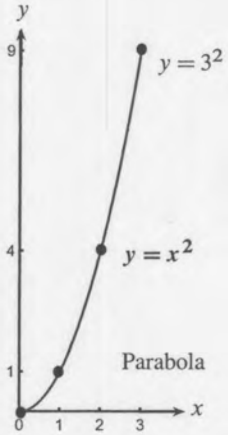
I believe that seeing examples of functions, and using the word to explain those examples, is a fast and powerful way to learn. I will start with three examples:

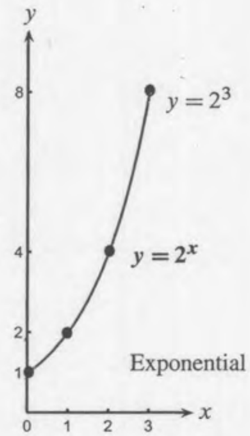
Linear function	$y(x) = 2x$
Squaring function	$y(x) = x^2$
Exponential function	$y(x) = 2^x$

The first point is that those are not the same! Their formulas involve 2 and x in very different ways. When I draw their graphs (this is a good way to understand functions) you see that all three are increasing when x is positive. The slopes are positive.

When the input x increases (moving to the right), the output y also increases (the graph goes upward). The three functions increase at different *rates*.







Near the start at $x=0$, the first function increases the fastest. But the others soon catch up. All three graphs reach the same height $y=4$ when $x=2$. Beyond that point the second graph $y=x^2$ pulls ahead. At $x=3$ the squaring function reaches $y=3^2=9$, while the height of the third graph is only $y=2^3=8$.

Don't be deceived, *the exponential will win*. It pulls even at $x=4$, because 4^2 and 2^4 are both 16. Then $y=2^x$ moves ahead of $y=x^2$ and it stays ahead. When you reach $x=10$, the third graph will have $y=2^{10}=1024$ compared to $y=10^2=100$.

The graphs themselves are a *straight line* and a *parabola* and an *exponential*. The straight line has constant growth rate. The parabola has increasing growth rate. The exponential curve has exponentially increasing growth rate. I emphasize these because calculus is all about growth rates.

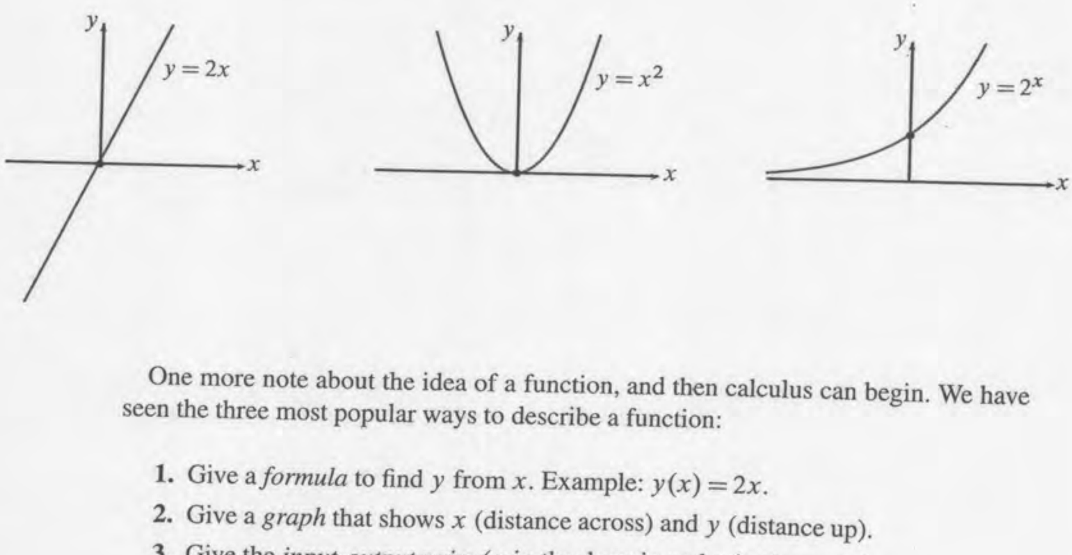
The whole point of differential calculus is to discover the growth rate of a function, and to use that information. So there are actually **two functions** in play at the same time—*the original function and its growth rate*. Before I go further down this all-important road, let me give a working definition of a function $y(x)$:

A function has inputs x and outputs $y(x)$. To each x it assigns one y.

The inputs x come from the "domain" of the function. In our graphs the domain contained all numbers $x \geqslant 0$. The outputs y form the "range" of the function. The ranges for the first two functions $y=2x$ and $y=x^2$ contained all numbers $y \geqslant 0$. But the range for $y=2^x$ is limited to $y \geqslant 1$ when the domain is $x \geqslant 0$.

Since these examples are so important, let me also allow x to be *negative*. The three graphs are shown below. Strictly speaking, these are new functions! Their domains have been extended to *all real numbers* x. Notice that the three ranges are also different:

The range of $y=2x$ is all real numbers y
The range of $y=x^2$ is all nonnegative numbers $y \geqslant 0$
The range of $y=2^x$ is all positive numbers $y>0$



One more note about the idea of a function, and then calculus can begin. We have seen the three most popular ways to describe a function:

1. Give a *formula* to find y from x. Example: $y(x)=2x$.
2. Give a *graph* that shows x (distance across) and y (distance up).
3. Give the *input-output pairs* (x in the domain and y in the range).

In a high-level definition, the "function" is the set of all the input-output pairs. We could also say: The function is the rule that assigns an output y in the range to every input x in the domain.

This shows something that we see for other words too. Logically, the definition should come first. Practically, we understand the definition better after we know examples that use the word. Probably that is the way we learn other words and also the way we will learn calculus. Examples show the general idea, and the definition is more precise. Together, we get it right.

The first words in this book were *Calculus is about functions*. Now I have to update that.

PAIRS OF FUNCTIONS

Calculus is about pairs of functions. Call them Function (1) and Function (2). Our graphs of $y=2x$ and $y=x^2$ and $y=2^x$ were intended to be examples of Function (1). Then we discussed the growth rates of those three examples. **The growth rate of Function (1) is Function (2)**. This is our first task—to find the growth rate of a function. Differential calculus starts with a formula for Function (1) and aims to produce a formula for Function (2).

Let me say right away how calculus operates. There are two ways to compute how quickly y changes when x changes:

Method 1 (*Limits*): Write $\dfrac{\text{Change in } y}{\text{Change in } x} = \dfrac{\Delta y}{\Delta x}$. Take the limit of this ratio as $\Delta x \to 0$.

Method 2 (*Rules*): Follow a rule to produce new growth rates from known rates.

For each new function $y(x)$, look to see if it can be produced from known functions—obeying one of the rules. An important part of learning calculus is to see different ways of producing new functions from old. Then we follow the rules for the growth rate.

Suppose the new function is *not* produced from known functions (2^x is not produced from $2x$ or x^2). Then we have to find its growth rate directly. By "directly" I mean that we compute a limit which is Function (2). This book will explain what a "limit" means and how to compute it.

Here we begin with examples—almost always the best way. I will state the growth rates "dy/dx" for the three functions we are working with:

Function (1)	$y=2x$	$y=x^2$	$y=2^x$
Function (2)	$\dfrac{dy}{dx}=2$	$\dfrac{dy}{dx}=2x$	$\dfrac{dy}{dx}=2^x(\ln 2)$

The linear function $y=2x$ has constant growth rate $dy/dx=2$. This section will take that first and easiest step. It is our opportunity to connect the growth rate to the **slope of the graph**. The ratio of *up* to *across* is $2x/x$ which is 2.

Section 0.2 takes the next step. The squaring function $y=x^2$ has linear growth rate $dy/dx=2x$. (This requires the idea of a limit—so fundamental to calculus.) Then we can introduce our first two rules:

Constant factor The growth rate of $Cy(x)$ is C times the growth rate of $y(x)$.

Sum of functions The growth rate of y_1+y_2 is the sum of the two growth rates.

The first rule says that $y = 5x^2$ has growth rate $10x$. The factor $C = 5$ multiplies the growth rate $2x$. The second rule says that $y_1 + y_2 = 5x^2 + 2x$ has growth rate $10x + 2$. Notice how we immediately took $5x^2$ as a function y_1 with a known growth rate. Together, the two rules give the growth rate for any "linear combination" of y_1 and y_2:

The growth rate of $C_1 y_1 + C_2 y_2$ **is that same combination** $C_1 \dfrac{dy_1}{dx} + C_2 \dfrac{dy_2}{dx}$.

In words, the step from Function (1) to Function (2) is *linear*. The slope of $y = x^2 - x$ is $dy/dx = 2x - 1$. This rule is simple but so important.

Finally, Section 0.3 will present the exponential functions $y = 2^x$ and $y = e^x$. Our first job is their meaning—what is "2 *to the power* π"? We understand $2^3 = 8$ and $2^4 = 16$, but how can we multiply 2 by itself π times?

When we meet e^x, we are seeing the great creation of calculus. This is a function with the remarkable property that $dy/dx = y$. **The slope equals the function.** This requires the amazing number e, which was never seen in algebra—because it only appears when you take the right limit.

So these first sections compute growth rates for three essential functions. We are ready for $y = 2x$.

THE SLOPE OF A GRAPH

The slope is distance up divided by distance across. I am thinking now about the graph of a function $y(x)$. The "distance across" is the change $x_2 - x_1$ in the inputs, from x_1 to x_2. The "distance up" is the change $y_2 - y_1$ in the outputs, from y_1 to y_2. The slope is large and the graph is steep when $y_2 - y_1$ is much larger than $x_2 - x_1$. *Change in y divided by change in x* matches our ordinary meaning of the word slope:

$$\textbf{Average slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}. \tag{1}$$

I introduced the very useful Greek letter Δ (delta), as a symbol for *change*. We take a step of length Δx to go from x_1 to x_2. For the height $y(x)$ on the graph, that produces a step $\Delta y = y_2 - y_1$. The ratio of Δy to Δx, up divided by across, is the average slope between x_1 and x_2. The slope is the steepness.

Important point: I had to say "average" because the slope could be changing as we go. The graph of $y = x^2$ shows an increasing slope. Between $x_1 = 1$ and $x_2 = 2$, what is the average slope for $y = x^2$? *Divide* Δy *by* Δx:

$$\begin{array}{l} y_1 = 1 \text{ at } x_1 = 1 \\ y_2 = 4 \text{ at } x_2 = 2 \end{array} \qquad \textbf{Average slope} = \frac{4-1}{2-1} = \frac{\Delta y}{\Delta x} = 3.$$

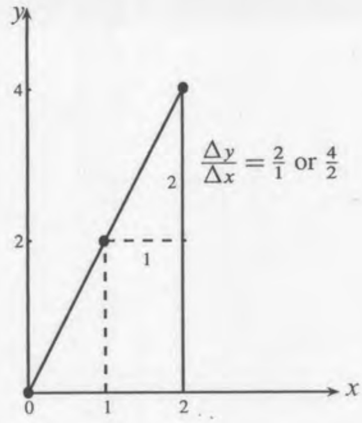
Between $x_1 = 0$ and $x_2 = 2$, we get a different answer (not 3). This graph of x^2 shows the problem of calculus, to deal with changes in slope and changes in speed.

The graph of $y = 2x$ has constant slope. The ratio of Δy to Δx, distance up to distance across, is always 2:

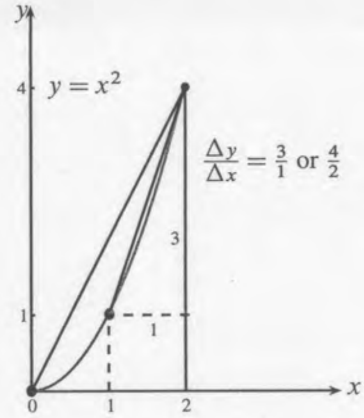
$$\textbf{Constant slope} \qquad \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2x_2 - 2x_1}{x_2 - x_1} = 2.$$

The mathematics is easy, which gives me a chance to emphasize the words and the ideas:

Function (1) = **Height** of the graph   Function (2) = **Slope** of the graph



Constant slope



Changing slope

When Function (1) is $y = Cx$, the ratio $\Delta y / \Delta x$ is always C. A linear function has a constant slope. And those same functions can come from driving a car at constant speed:

Function (1) = **Distance** traveled = $\boldsymbol{Ct}$ Function (2) = **Speed** of the car = $\boldsymbol{C}$

For a graph of Function (1), its rate of change is the **slope**. When Function (1) measures distance traveled, its rate of change is the **speed** (or **velocity**). When Function (1) measures our height, its rate of change is our **growth rate**.

The first point is that *functions are everywhere*. For calculus, those functions come in pairs. *Function* (2) *is the rate of change of Function* (1).

The second point is that Function (1) and Function (2) are measured in different units. That is natural:

$$\left(\text{Speed in } \frac{\textbf{miles}}{\textbf{hour}}\right) \text{ multiplies } \left(\text{Time in } \textbf{hours}\right) \text{ to give } \left(\text{Distance in } \textbf{miles}\right)$$

$$\left(\text{Growth rate in } \frac{\textbf{inches}}{\textbf{year}}\right) \text{ multiplies } \left(\text{Time in } \textbf{years}\right) \text{ to give } \left(\text{Height in } \textbf{inches}\right)$$

When time is in seconds and distance is in meters, then speed is automatically in meters per second. We can choose two units, and they decide the third. Function (2) always involves a division: Δy is divided by Δx or distance is divided by time.

The delicate and tricky part of calculus is coming next. We want the ***slope at one point*** and the ***speed at one instant***. What is the rate of change in *zero time*?

The distance across is $\Delta x = 0$ at a point. The distance up is $\Delta y = 0$. **Formally, their ratio is** $\frac{0}{0}$. It is the inspiration of calculus to give this a useful meaning.

Big Picture

Calculus connects Function (1) with Function (2) = **rate of change** of (1)

Function (1) Distance traveled $f(t)$ Function (2) Speed $s(t) = df/dt$

Function (1) Height of graph $y(x)$ Function (2) Slope $s(x) = dy/dx$

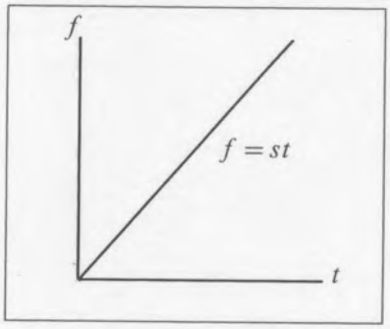
Function (2) tells how quickly Function (1) is changing

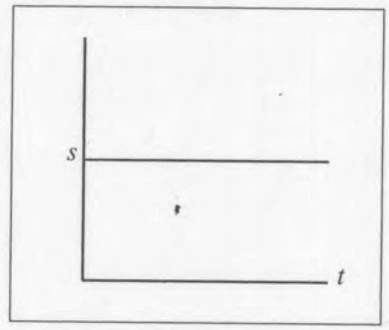
KEY Constant speed $s = \dfrac{\text{Distance } f}{\text{Time } t}$ Constant slope $s = \dfrac{\text{Distance up}}{\text{Distance across}}$

Graphs of (1) and (2)

f = increasing distance

s = constant speed





Slope of f-graph $= \dfrac{\text{up}}{\text{across}} = \dfrac{st}{t} = s$

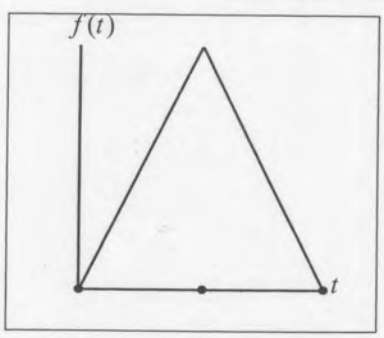
Area under s-graph = area of rectangle $= st = f$

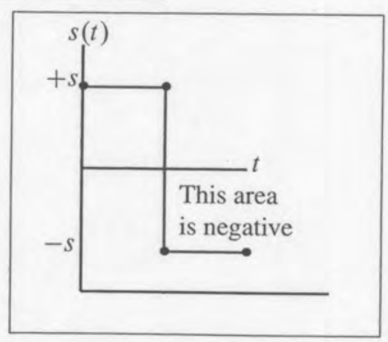
Now run the car backwards.

Speed is negative

Distance goes down to 0

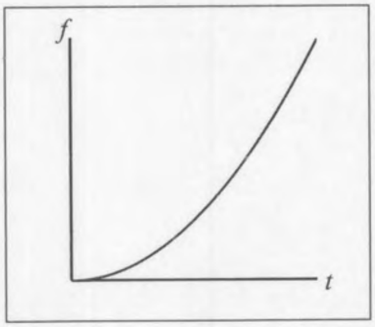
Area "under" $s(t)$ is zero

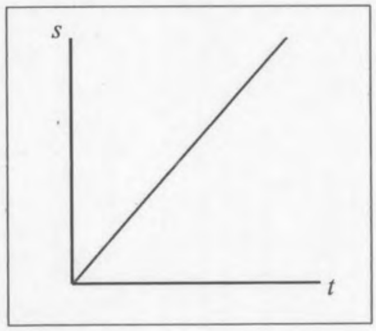




Example with increasing speed Then distance has steeper slope

$$\boxed{\begin{aligned} f &= 10t^2 \\ s &= 20t \end{aligned}}$$





When speed is changing, algebra is not enough $s = \dfrac{f}{t}$ is wrong

Still true that area under s = triangle area $= \dfrac{1}{2}(t)(20t) = 10t^2 = f$

Still true that s = slope of $f = \dfrac{df}{dt}$ = "derivative" of f

When f is increasing, the slope s is **positive**

When f is decreasing, the slope s is **negative**

When f is at its maximum or minimum, the slope s is **zero**

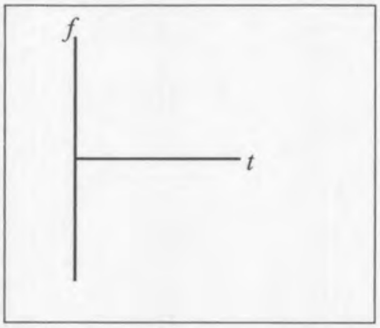
The graphs of any $f(t)$ and $f(t)+10$ have the same slope at every t

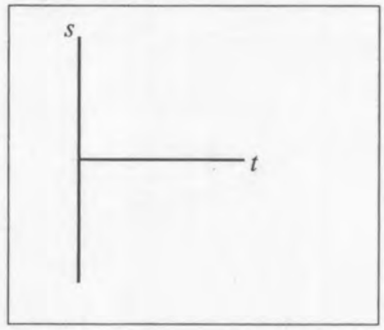
To recover f = Function (1) from $\dfrac{df}{dt}$, good to know a starting height $f(0)$

Practice Questions

1. Draw a graph of $f(t)$ that goes up and down and up again.

Then draw a reasonable graph of its slope.





2. The world population $f(t)$ increased slowly at first, now quickly, then slowly again (we hope and expect). Maybe a limit $\approx$ 12 or 14 billion.

Draw a graph for $f(t)$ and its slope $s(t) = \dfrac{df}{dt}$

3. Suppose $f(t) = 2t$ for $t \leq 1$ and then $f(t) = 3t + 2$ for $t \geq 1$

Describe the slope graph $s(t)$. Compare its area out to $t = 3$ with $f(3)$

4. Draw a graph of $f(t) = \cos t$. Then sketch a graph of its slope. At what angles t is the slope zero (slope $= 0$ when $f(t)$ is "flat").
5. Suppose the graph of $f(t)$ is shaped like the capital letter **W**. Describe the graph of its slope $s(t) = \dfrac{df}{dt}$. What is the total area under the graph of s ?
6. A train goes a distance f at constant speed s. Inside the train, a passenger walks forward a distance F at walking speed S. What distance does the passenger go? At what speed? (Measure distance from the train station)

0.2 The Changing Slope of $y = x^2$ and $y = x^n$

The second of our three examples is $y = x^2$. Now the slope is changing as we move up the curve. The average slope is still $\Delta y/\Delta x$, but that is not our final goal. We have to answer the crucial questions of differential calculus:

What is the meaning of "slope at a point" and how can we compute it ?

My video lecture on *Big Picture: Derivatives* also faces those questions. Every student of calculus soon reaches this same problem. What is the meaning of "rate of change" when we are at a single moment in time, and nothing actually changes in that moment ? Good question.

The answers will come in two steps. Algebra produces $\Delta y/\Delta x$, and then calculus finds dy/dx. Those steps dy and dx are infinitesimally short, so formally we are looking at $0/0$. Trying to define dy and dx and $0/0$ is not wise, and I won't do it. The successful plan is to realize that the ratio of Δy to Δx is clearly defined, and those two numbers can become very small. ***If that ratio $\Delta y/\Delta x$ approaches a limit, we have a perfect answer***:

$$\textbf{The slope at } x \textbf{ is the limit of} \qquad \frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x}.$$

The distance across, from x to $x + \Delta x$, is just Δx. The distance up is from $y(x)$ to $y(x + \Delta x)$. Let me show how algebra leads directly to $\Delta y/\Delta x$ when $y = x^2$:

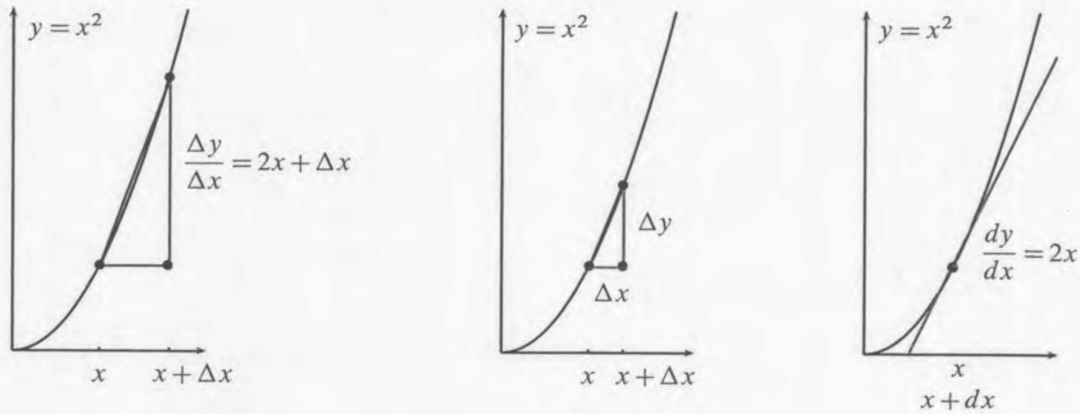
$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x.$$

Notice that calculation! The "leading terms" x^2 and $-x^2$ cancel. The important term here is $2x\Delta x$. This "first-order term" is responsible for most of Δy. The "second-order term" in this example is $(\Delta x)^2$. *After we divide by Δx, this term is still small.* That part $(\Delta x)^2/\Delta x$ will disappear as the step size Δx goes to zero.

That limiting process $\Delta x \to 0$ produces the slope dy/dx at a point. The first-order term survives in dy/dx and higher-order terms disappear.

$$\textbf{Slope at a point} \qquad \frac{dy}{dx} = \textbf{limit of } \frac{\Delta y}{\Delta x} = \textbf{limit of } 2x + \Delta x = 2x.$$

Algebra produced $\Delta y/\Delta x$. In the limit, calculus gave us dy/dx. Look at the graph, to see the geometry of those steps. The ratio up/across $= \Delta y/\Delta x$ is the slope between two points on the graph. *The two points come together in the limit.* Then $\Delta y/\Delta x$ approaches the slope dy/dx at a ***single point***.



The color lines connecting points on the first two graphs are called "chords." They approach the color line on the third graph, which touches at only *one* point. This is the "**tangent line**" to the curve. Here is the idea of differential calculus:

$$\textbf{Slope of tangent line} = \textbf{Slope of curve} = \text{ Function (2)} = \frac{dy}{dx} = \mathbf{2x}.$$

To find the equation for this tangent line, return to algebra. Choose any specific value x_0. Above that position on the x axis, the graph is at height $y_0 = x_0^2$. The slope of the tangent line at that point of the graph is $dy/dx = 2x_0$. **We want the equation for the line through that point with that slope**.

Equation for the tangent line $\quad y - y_0 = (2x_0)(x - x_0) \quad$ (1)

At the point where $x = x_0$ and $y = y_0$, this equation becomes $0 = 0$. The equation is satisfied and the point is on the line. Furthermore the slope of the line matches the slope $2x_0$ of the curve. You see that directly if you divide both sides by $x - x_0$:

$$\textbf{Tangent line} \qquad \frac{\text{up}}{\text{across}} = \frac{y - y_0}{x - x_0} = 2x_0 \text{ is the correct slope } \frac{dy}{dx}.$$

Let me say this again. The curve $y = x^2$ is bending, the tangent line is straight. This line stays as close to the curve as possible, near the point where they touch. The tangent line gives a *linear* approximation to the nonlinear function $y = x^2$:

Linear approximation $\quad y \approx y_0 + (2x_0)(x - x_0) = y_0 + \frac{dy}{dx}(x - x_0) \quad$ (2)

I only moved y_0 to the right side of equation (1). Then I used the symbol $\approx$ for "approximately equal" because the symbol $=$ would be wrong: The curve bends.

Important for the future: This bending comes from the ***second derivative*** of $y = x^2$.

THE SECOND DERIVATIVE

The first derivative is the slope $dy/dx = 2x$. **The second derivative is the slope of the slope**. By good luck we found the slope of $2x$ in the previous section (easy to do, it is just the constant 2). Notice the symbol d^2y/dx^2 for the slope of the slope:

Second derivative $\dfrac{d^2y}{dx^2}$ The slope of $\dfrac{dy}{dx} = 2x$ is $\dfrac{d^2y}{dx^2} = 2$. (3)

In ordinary language, the first derivative dy/dx tells how fast the function $y(x)$ is changing. The second derivative tells whether we are *speeding up or slowing down*. The example $y = x^2$ is certainly speeding up, since the graph is getting steeper. The curve is bending and the tangent line is steepening.

Think also about $y = x^2$ on the left side (the negative side) of $x = 0$. The graph is coming down to zero. Its slope is certainly negative. But the curve is still bending upwards! The algebra agrees with this picture: The slope $dy/dx = 2x$ is *negative* on the left side of $x = 0$, but the second derivative $d^2y/dx^2 = 2$ is still positive.

An economist or an investor watches all three of those numbers: $y(x)$ tells where the economy is, and dy/dx tells which way it is going (short term, close to the tangent line). But it is d^2y/dx^2 that reveals the longer term prediction. I am writing these words near the end of the economic downturn (I hope). I am sorry that dy/dx has been negative but happy that d^2y/dx^2 has recently been positive.

DISTANCE AND SPEED AND ACCELERATION

An excellent example of $y(x)$ and dy/dx and d^2y/dx^2 comes from driving a car. The function y is the *distance traveled*. Its rate of change (first derivative) is the *speed*. The rate of change of the speed (second derivative) is the *acceleration*. If you are pressing on the gas pedal, all three will be positive. If you are pressing on the brake, the distance and speed are probably still positive but the acceleration is negative: The speed is dropping. If the car is *in reverse* and you are *braking*, what then?

The speed is negative (going backwards)
The speed is increasing (less negative)
The acceleration is positive (increasing speed).

The video lecture mentions that car makers don't know calculus. The distance meter on the dashboard does not go back toward zero (in reverse gear it should). The speedometer does not go below zero (it should). There is no meter at all (on my car) for acceleration. Spaceships do have accelerometers, and probably aircraft too.

We often hear that an astronaut or a test pilot is subjected to a high number of g's. The ordinary acceleration in free fall is one g, from the gravity of the Earth. An airplane in a dive and a rocket at takeoff will have a high second derivative—the rocket may be hardly moving but it is accelerating like mad.

One more very useful point about this example of motion. *The natural letter to use is not x but t*. The distance is a function of **time**. The slope of a graph is up/across, but now the right ratio is (change of distance) divided by (change in time):

Average speed between t and $t + \Delta t$ $\quad \dfrac{\Delta y}{\Delta t} = \dfrac{y(t+\Delta t) - y(t)}{\Delta t}$

Speed at t itself (instant speed) $\quad \dfrac{dy}{dt} = \text{limit of } \dfrac{\Delta y}{\Delta t} \text{ as } \Delta t \to 0$

The words "rate of change" and "rate of growth" suggest t. The word "slope" suggests x. But calculus doesn't worry much about the letters we use. If we graph the distance traveled as a function of time, then the x axis (across) becomes the t axis. And the slope of that graph becomes the speed (velocity is the best word).

Here is something not often seen in calculus books—the **second difference**. We know the first difference $\Delta y = y(t+\Delta t) - y(t)$. It is the change in y. The second difference $\boldsymbol{\Delta^2 y}$ **is the change in** $\boldsymbol{\Delta y}$:

Second difference $$\Delta^2 y = (y(t+\Delta t) - y(t)) - (y(t) - y(t-\Delta t)) \qquad \frac{\boldsymbol{\Delta^2 y}}{\boldsymbol{(\Delta t)^2}} \to \frac{\boldsymbol{d^2 y}}{\boldsymbol{dt^2}} \tag{4}$$

$\Delta^2 y$ simplifies to $y(t+\Delta t) - 2y(t) + y(t-\Delta t)$. We divide by $(\Delta t)^2$ to approximate the acceleration. In the limit as $\Delta t \to 0$, this ratio $\Delta^2 y/(\Delta t)^2$ becomes the **second derivative** $\boldsymbol{d^2 y/dt^2}$.

THE SLOPE OF $y = x^n$

The slope of $y = x^2$ is $dy/dx = 2x$. Now I want to compute the slopes of $y = x^3$ and $y = x^4$ and all succeeding powers $y = x^n$. The rate of increase of x^n will be found again in Section 2.2. But there are two reasons to discover these special derivatives early:

1. Their pattern is simple: **The slope of each power** $\boldsymbol{y = x^n}$ **is** $\boldsymbol{\dfrac{dy}{dx} = nx^{n-1}}$.
2. The next section can then introduce $y = e^x$. This amazing function has $\dfrac{dy}{dx} = y$.

Of course $y = x^2$ fits into this pattern for x^n. The exponent drops by 1 from $n = 2$ to $n - 1 = 1$. Also $n = 2$ multiplies that lower power to give $nx^{n-1} = 2x$.

The slope of $\boldsymbol{y = x^3}$ ***is*** $\boldsymbol{dy/dx = 3x^2}$. Watch how $3x^2$ appears in $\Delta y/\Delta x$:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}. \tag{5}$$

Cancel x^3 with $-x^3$. Then divide by Δx:

Average slope $$\frac{\Delta y}{\Delta x} = \mathbf{3x^2} + 3x\Delta x + (\Delta x)^2.$$

When the step length Δx goes to zero, the limit value dy/dx is $3x^2$. This is nx^{n-1}.

To establish this pattern for $n = 4, 5, 6, \ldots$ the only hard part is $(x + \Delta x)^n$. When n was 3, we multiplied this out in equation (5) above. The result will always start with x^n. We claim that the next term (the "first-order term" in Δy) will be $nx^{n-1}\Delta x$. When we divide this part of Δy by Δx, we have the answer we want—the correct derivative nx^{n-1} of $y(x) = x^n$.

How to see that term $nx^{n-1}\Delta x$? Our multiplications showed that $2x\Delta x$ and $3x^2\Delta x$ are correct for $n = 2$ and 3. Then we can reach $n = 4$ from $n = 3$:

$$\begin{aligned}(x+\Delta x)^4 &= (x+\Delta x)^3 \text{ times } (x+\Delta x)\\ &= (x^3 + 3x^2\Delta x + \cdots) \text{ times } (x+\Delta x)\end{aligned}$$

That multiplication produces x^4 and $4x^3\Delta x$, exactly what we want. We can go from each n to the next one in the same way (this is called "induction"). The derivatives of all the powers $x^4, x^5, \ldots, x^n$ are $4x^3, 5x^4, \ldots, nx^{n-1}$.

Section 2.2 of the book shows you a slightly different proof of this formula. And the video lecture on the *Product Rule* explains one more way. Look at x^{n+1} as the product of x^n times x, and use the rule for the slope of y_1 times y_2:

Product Rule **Slope of $y_1 y_2 = y_2$ (slope of y_1) + y_1 (slope of y_2)**	(6)

With $y_1 = x^n$ and $y_2 = x$, the slope of $y_1 y_2 = x^{n+1}$ comes out right:

$$x\,(\text{slope of } x^n) + x^n\,(\text{slope of } x) = x(nx^{n-1}) + x^n(1) = \boldsymbol{(n+1)x^n}. \qquad (7)$$

Again we can increase n one step at a time. Soon comes the truly valuable fact that this derivative formula is correct for *all powers* $y = x^n$. The exponent n can be negative, or a fraction, or any number at all. The slope dy/dx is always nx^{n-1}.

By combining different powers of x, you know the slope of every "polynomial." An example is $y = x + x^2/2 + x^3/3$. Compute dy/dx one term at a time, as the Sum Rule allows:

$$\frac{d}{dx}\left(x + \frac{x^2}{2} + \frac{x^3}{3}\right) = 1 + x + x^2.$$

The slope of the slope is $d^2y/dx^2 = 1 + 2x$. The fourth derivative is zero!

Function (1) tells us the height y above each point x

The problem is to find the "instant slope" at x

This slope $s(x)$ is written $\dfrac{dy}{dx}$ It is **Function (2)**

KEY: $\dfrac{\boldsymbol{\Delta y}}{\boldsymbol{\Delta x}} = \dfrac{y(x+\Delta x) - y(x)}{\Delta x} = \dfrac{\text{up}}{\text{across}}$ approaches $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}}$ as $\Delta x \to 0$

Compute the **instant slope** $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}}$ for the function $y = x^3$

First find the average slope between x and $x + \Delta x$

Average slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{(x+\Delta x)^3 - x^3}{\Delta x}$

Write $(x+\Delta x)^3 = x^3 + \boldsymbol{3x^2\Delta x} + 3x(\Delta x)^2 + (\Delta x)^3$

Subtract x^3 and divide by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\boldsymbol{3x^2\Delta x} + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \boldsymbol{3x^2} + 3x\Delta x + (\Delta x)^2$$

When $\Delta x \to 0$, this becomes $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}} = \boldsymbol{3x^2}$ $\qquad \dfrac{d}{dx}(x^n) = nx^{n-1}$

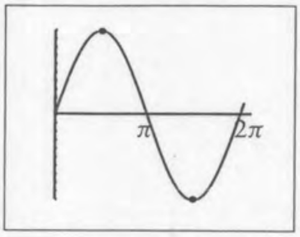
$y = Cx^n$ has slope Cnx^{n-1} The slope of $y = 7x^2$ is $\dfrac{dy}{dx} = 14x$

Multiply y by $C \to$ Multiply Δy by $C \to$ Multiply $\dfrac{dy}{dx}$ by C

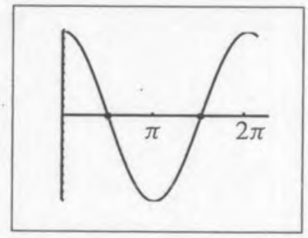
Neat Fact: The slope of $\mathbf{y = \sin x}$ is $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{\cos x}$

The graphs show this is reasonable

Slope at the start is 1 (to find later)



$y = \sin x$
slope $= \cos x$



Sine curve climbing → Cosine curve > 0

Top of sine curve (flat) → Cosine is zero at the first bullet

Sine curve falling → Cosine curve < 0 between bullets

Bottom of sine curve (flat) → Cosine back to zero at the second bullet

Practice Questions

1. For $y = 2x^3$, what is the average slope $= \dfrac{\Delta y}{\Delta x}$ from $x = 1$ to $x = 2$?

2. What is the instant slope of $y = 2x^3$ at $x = 1$? What is $\dfrac{d^2y}{dx^2}$?

3. $y = x^n$ has $\dfrac{dy}{dx} = nx^{n-1}$. What is $\dfrac{dy}{dx}$ when $y(x) = \dfrac{1}{x} = x^{-1}$?

4. For $y = x^{-1}$, what is the average slope $\dfrac{\Delta y}{\Delta x}$ from $x = \dfrac{1}{2}$ to $x = 1$?

5. What is the instant slope of $y = x^{-1}$ at $x = \dfrac{1}{2}$?

6. Suppose the graph of $y(x)$ climbs up to its maximum at $x = 1$

Then it goes downward for $x > 1$

6A. What is the sign of $\dfrac{dy}{dx}$ for $x < 1$ and then for $x > 1$?

6B. What is the instant slope at $x = 1$?

7 If $y = \sin x$, write an expression for $\dfrac{\Delta y}{\Delta x}$ at any point x.

We see later that this $\dfrac{\Delta y}{\Delta x}$ approaches $\cos x$

0.3 The Exponential $y = e^x$

The great function that calculus creates is the exponential $y = e^x$. There are different ways to reach this function, and Section 6.2 of this textbook mentions five ways. Here I will describe the approach to e^x that I now like best. It uses the derivative of x^n, the first thing we learn.

In all approaches, a "limiting step" will be involved. So the amazing number $e = 2.7\ldots$ is not seen in algebra (e is not a fraction). The question is where to take that limiting step, and my answer starts with this truly remarkable fact: When $y = e^x$ is **Function (1), it is also Function (2)**.

$$\textit{The exponential function } y = e^x \textit{ solves the equation } \frac{dy}{dx} = y.$$

The function equals its slope. This is a first example of a **differential equation**—connecting an unknown function y with its own derivatives. Fortunately $dy/dx = y$ is the most important differential equation—a model that other equations try to follow.

I will add one more requirement, to eliminate solutions like $y = 2e^x$ and $y = 8e^x$. When $y = e^x$ solves our equation, all other functions Ce^x solve it too. ($C = 2$ and $C = 8$ will appear on both sides of $dy/dx = y$, and they cancel.) At $x = 0$, e^0 will be the "zeroth power" of the positive number e. *All zeroth powers are* 1. So we want $y = e^x$ to equal 1 when $x = 0$:

$$y = e^x \textit{ is the solution of } \frac{dy}{dx} = y \textit{ that starts from } y = 1 \textit{ at } x = 0.$$

Before solving $dy/dx = y$, look at what this equation means. When y starts from 1 at $x = 0$, its slope is also 1. So y increases. Therefore dy/dx also increases, staying equal to y. So y increases faster. The graph gets steeper as the function climbs higher. This is what "growing exponentially" means.

INTRODUCING e^x

Exponential growth is quite ordinary and reasonable. When a bank pays interest on your money, the interest is proportional to the amount you have. After the interest is added, you have more. The new interest is based on the higher amount. Your wealth is growing "geometrically," one step at a time.

At the end of this section on e^x, I will come back to *continuous* compounding—interest is added at every instant instead of every year. That word "continuous" signals that we need calculus. There is a limiting step, from every year or month or day or second to every instant. You don't get infinite interest, you do get exponentially increasing interest.

I will also describe other ways to introduce e^x. This is an important question with many answers! I like equation (1) below, because we know the derivative of each power x^n. If you take their derivatives in equation (1), you get back the same e^x: *amazing*. So that sum solves $dy/dx = y$, starting from $y = 1$ as we wanted.

The difficulty is that the sum involves every power x^n: an *infinite series*. When I go step by step, you will see that those powers are all needed. For this infinite series, I am asking you to believe that everything works. *We can add the series to get* e^x, and we can add all derivatives to see that the slope of e^x is e^x.

For me, the advantage of using only the powers x^n is overwhelming.

CONSTRUCTING $y = e^x$

I will solve $dy/dx = y$ a step at a time. At the start, $y = 1$ means that $dy/dx = 1$:

$$\textbf{Start}\quad \begin{array}{r} y=1 \\ dy/dx=1 \end{array} \qquad \textbf{Change } y \quad \begin{array}{r} y=1+x \\ dy/dx=1 \end{array} \qquad \textbf{Change } \frac{dy}{dx} \quad \begin{array}{r} y=1+x \\ dy/dx=1+x \end{array}$$

After the first change, $y = 1 + x$ has the correct derivative $dy/dx = 1$. But then I had to change dy/dx to keep it equal to y. And I can't stop there:

$$\begin{array}{r} y=1+x \\ dy/dx=1+x \end{array} \qquad \textbf{Update } y \textbf{ to } 1+x+\frac{1}{2}x^2 \quad \textbf{Then update } \frac{dy}{dx} \textbf{ to } 1+x+\frac{1}{2}x^2$$

The extra $\frac{1}{2}x^2$ gave the correct x in the slope. Then $\frac{1}{2}x^2$ also had to go into dy/dx, to keep it equal to y. Now we need a new term with this derivative $\frac{1}{2}x^2$.

The term that gives $\frac{1}{2}x^2$ has x^3 divided by 6. The derivative of x^n is nx^{n-1}, so I *must divide by* n (to cancel correctly). Then the derivative of $x^3/6$ is $3x^2/6 = \frac{1}{2}x^2$ as we wanted. After that comes x^4 divided by 24:

$$\begin{array}{lll} \dfrac{x^3}{6} = \dfrac{x^3}{(3)(2)(1)} & \text{has slope} & \dfrac{x^2}{(2)(1)} \\ \dfrac{x^4}{24} = \dfrac{x^4}{(4)(3)(2)(1)} & \text{has slope} & \dfrac{4x^3}{(4)(3)(2)(1)} = \dfrac{x^3}{6}. \end{array}$$

The pattern becomes more clear. The x^n term is divided by *n factorial*, which is $n! = (n)(n-1)\ldots(1)$. The first five factorials are $1, 2, 6, 24, 120$. **The derivative of that term $x^n/n!$ is the previous term** $x^{n-1}/(n-1)!$ (because the n's cancel). As long as we don't stop, this sum of infinitely many terms does achieve $dy/dx = y$:

$$\boldsymbol{y(x) = e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots + \tfrac{1}{n!}x^n + \cdots} \qquad (1)$$

If we substitute $x = 10$ into this series, do the infinitely many terms add to a finite number e^{10}? *Yes*. The numbers $n!$ grow much faster than 10^n (or any other x^n). So the terms $x^n/n!$ in this "exponential series" become extremely small as $n \to \infty$. Analysis shows that the sum of the series (which is $y = e^x$) does achieve $dy/dx = y$.

Note 1 Let me just remember a series that you know, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. If I replace $\frac{1}{2}$ by x, this becomes the *geometric series* $1 + x + x^2 + x^3 + \cdots$ and it adds up to $1/(1-x)$. This is the most important series in mathematics, but it runs into a problem at $x = 1$: the infinite sum $1 + 1 + 1 + 1 + \cdots$ doesn't "converge."

I emphasize that the series for e^x is always safe, because the powers x^n are divided by the rapidly growing numbers $n! = n$ *factorial*. This is a great example to meet, long before you learn more about convergence and divergence.

Note 2 Here is another way to look at that series for e^x. Start with x^n and take its derivative n times. First get nx^{n-1} and then $n(n-1)x^{n-2}$. Finally the nth derivative is $n(n-1)(n-2)\ldots(1)x^0$, which is *n factorial*. When we divide by that number, **the nth derivative of $x^n/n!$ is equal to 1**.

Now look at e^x. All its derivatives are still e^x. They all equal 1 at $x = 0$. *The series is matching every derivative of the function e^x at the starting point $x = 0$.*

Set $x = 1$ *in the exponential series*. This tells us the amazing number $e^1 = e$:

The number e	$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$	(2)

The first three terms add to 2.5. The first five terms almost reach 2.71. *We never reach* 2.72. With quite a few terms (how many ?) you can pass 2.71828. It is certain that e is not a fraction. It never appears in algebra, but it is the key number for calculus.

MULTIPLYING BY ADDING EXPONENTS

We write e^2 in the same way that we write 3^2. *Is it true that e times e equals* e^2 ? Up to now, e and e^2 come from setting $x = 1$ and $x = 2$ in the infinite series. The wonderful fact is that for every x, the series produces the "xth power of the number e." When $x = -1$, we get e^{-1} which is $1/e$:

Set $x = -1$ $$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$$

If we multiply that series for $1/e$ by the series for e, we get 1.

The best way is to go straight for all multiplications of e^x times any power e^X. The rule of adding exponents says that the answer is e^{x+X}. The series must say this too! When $x = 1$ and $X = -1$, this rule produces e^0 from e^1 times e^{-1}.

Add the exponents	$(e^x)(e^X) = e^{x+X}$	(3)

We only know e^x and e^X from the infinite series. For this all-important rule, we can multiply those series and recognize the answer as the series for e^{x+X}. Make a start:

Multiply each term
e^x times e^X
Hoping for
e^{x+X}

$$\begin{array}{l} e^x = \mathbf{1} + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ e^X = \mathbf{1} + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots \\ \hline (e^x)(e^X) = \mathbf{1} + x + X \quad + \frac{1}{2}x^2 + xX + \frac{1}{2}X^2 + \cdots \end{array} \qquad (4)$$

Certainly you see $x + X$. Do you see $\frac{1}{2}(x+X)^2$ in equation (4) ? No problem:

$$\frac{1}{2}(x+X)^2 = \frac{1}{2}(x^2 + 2xX + X^2) \quad \text{matches the "second degree" terms.}$$

The step to third degree takes a little longer, but it also succeeds:

$$\frac{1}{6}(x+X)^3 = \frac{1}{6}x^3 + \frac{3}{6}x^2X + \frac{3}{6}xX^2 + \frac{1}{6}X^3 \quad \text{matches the next terms in (4).}$$

For high powers of $x + X$ we need the *binomial theorem* (or a healthy trust that mathematics comes out right). When e^x multiplies e^X, the coefficient of $x^n X^m$ will be $1/n!$ times $1/m!$. Now look for that same term in the series for e^{x+X}:

$$\frac{(x+X)^{n+m}}{(n+m)!} \text{ includes } \frac{x^n X^m}{(n+m)!} \text{ times } \frac{(n+m)!}{n!m!} \text{ which gives } \frac{x^n X^m}{n!m!}. \qquad (5)$$

That binomial number $(n+m)!/n!\,m!$ is known to successful gamblers. It counts the number of ways to choose n aces out of $n+m$ aces. Out of 4 aces, you could choose 2 aces in $4!/2!2! = 6$ ways. To a mathematician, there are 6 ways to choose 2 x's out of $xxxx$. This number 6 will be the coefficient of x^2X^2 in $(x+X)^4$.

That 6 shows up in the fourth degree term. It is divided by 4! (to produce 1/4). When e^x multiplies e^X, $\frac{1}{2}x^2$ multiplies $\frac{1}{2}X^2$ (which also produces 1/4). All terms are good, but we are not going there—we accept $(e^x)(e^X) = e^{x+X}$ as now confirmed.

Note A different way to see this rule for $(e^x)(e^X)$ is based on $dy/dx = y$. Starting from $y = 1$ at $x = 0$, follow this equation. At the point x, you reach $y = e^x$. Now go an additional distance X to arrive at e^{x+X}.

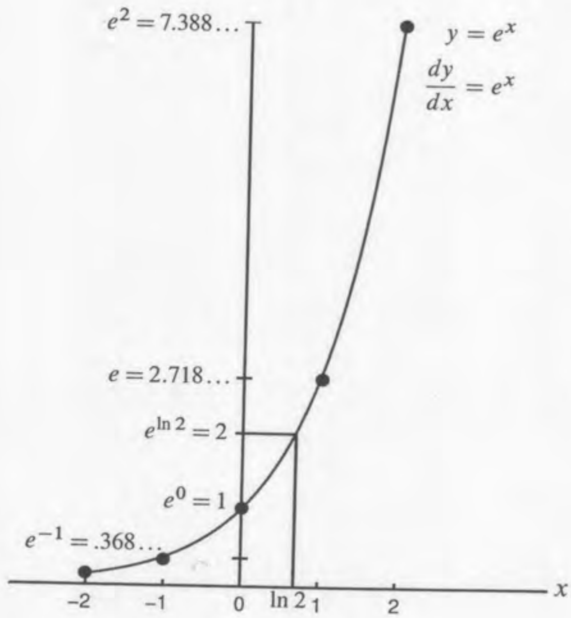
Notice that the additional part starts from e^x (instead of starting from 1). That starting value e^x will multiply e^X in the additional part. So e^x times e^X must be the same as e^{x+X}. (This is a "differential equations proof" that the exponents are added. Personally, I was happy to multiply the series and match the terms.)

The rule immediately gives e^x times e^x. The answer is $e^{x+x} = e^{2x}$. If we multiply again by e^x, we find $(e^x)^3$. This is equal to $e^{2x+x} = e^{3x}$. We are finding a new rule for all powers $(e^x)^n = (e^x)(e^x)\cdots(e^x)$:

Multiply exponents $\qquad (e^x)^n = e^{nx} \qquad (6)$

This is easy to see for $n = 1, 2, 3, \ldots$ and then $n = -1, -2, -3, \ldots$ *It remains true for all numbers x and n.*

That last sentence about "all numbers" is important! Calculus cannot develop properly without working with all exponents (not just whole numbers or fractions). The infinite series (1) defines e^x for every x and we are on our way. Here is the graph: **Function (1) = Function (2) = e^x = exp(x).**



$$(e^x)(e^X) = e^{x+X}$$
$$(e^x)^n = e^{nx}$$
$$e^{\ln y} = y$$

THE EXPONENTIALS 2^x AND b^x

We know that $2^3 = 8$ and $2^4 = 16$. But what is the meaning of 2^π ? One way to get close to that number is to replace π by 3.14 which is 314/100. As long as we have a fraction in the exponent, we can live without calculus:

Fractional power $\quad 2^{314/100} = $ 314th power of the 100th root $2^{1/100}$.

But this is only "close" to 2^π. And in calculus, we will want the slope of the curve $y = 2^x$. The good way is to connect 2^x with e^x, whose slope we know (it is e^x again). So we need to connect 2 with e.

The key number is the **logarithm of 2**. This is written "ln 2" and it is the power of e that produces 2. It is specially marked on the graph of e^x:

Natural logarithm of 2 $\qquad e^{\ln 2} = 2$

This number ln 2 is about 7/10. A calculator knows it with much higher accuracy. In the graph of $y = e^x$, the number ln 2 on the x axis produces $y = 2$ on the y axis.

This is an example where we want the output $y = 2$ and we ask for the input $x = \ln 2$. That is the opposite of knowing x and asking for y. "The logarithm $x = \ln y$ is the *inverse* of the exponential $y = e^x$." This idea will be explained in Section 4.3 and in two video lectures—inverse functions are not always simple.

Now 2^x has a meaning for every x. When we have the number ln 2, meeting the requirement $2 = e^{\ln 2}$, we can take the xth power of both sides:

Powers of 2 from powers of e $\qquad 2 = e^{\ln 2}$ and $\mathbf{2^x = e^{x \ln 2}}$. $\qquad$ (7)

All powers of e are defined by the infinite series. The new function 2^x also grows exponentially, but not as fast as e^x (because 2 is smaller than e). Probably $y = 2^x$ could have the same graph as e^x, if I stretched out the x axis. That stretching multiplies the slope by the constant factor ln 2. Here is the algebra:

Slope of $y = 2^x$ $\qquad \dfrac{d}{dx} 2^x = \dfrac{d}{dx} e^{x \ln 2} = (\ln 2)\, e^{x \ln 2} = \mathbf{(\ln 2) 2^x}$.

For any positive number b, the same approach leads to the function $y = b^x$. First, find the natural logarithm ln b. This is the number (positive or negative) so that $b = e^{\ln b}$. Then take the xth power of both sides:

Connect b to e $\quad b = e^{\ln b}$ and $b^x = e^{x \ln b}$ and $\dfrac{d}{dx} b^x = (\ln b) b^x \qquad$ (8)

When b is e (the perfect choice), $\ln b = \ln e = 1$. When b is e^n, then $\ln b = \ln e^n = n$. "***The logarithm is the exponent.***" Thanks to the series that defines e^x for every x, that exponent can be any number at all.

Allow me to mention Euler's Great Formula $\boldsymbol{e^{ix} = \cos x + i \sin x}$. The exponent ix has become an **imaginary number**. (You know that $i^2 = -1$.) If we faithfully use $\cos x + i \sin x$ at 90° and 180° (where $x = \pi/2$ and $x = \pi$), we arrive at these amazing facts:

Imaginary exponents $\qquad e^{i\pi/2} = i$ and $\boldsymbol{e^{i\pi} = -1}$. $\qquad$ (9)

Those equations are not imaginary, they come from the great series for e^x.

CONTINUOUS COMPOUNDING OF INTEREST

There is a different and important way to reach e and e^x (not by an infinite series). We solve the key equation $dy/dx = y$ in small steps. As these steps approach zero (a limit is always involved !) the small-step solution becomes the exact $y = e^x$.

I can explain this idea in two different languages. Each step multiplies by $1 + \Delta x$:

1. *Compound interest.* After each step Δx, the interest is added to y. Then the next step begins with a larger amount, and y increases exponentially.
2. *Finite differences.* The continuous dy/dx is replaced by small steps $\Delta Y/\Delta x$:

$$\frac{dy}{dx} = y \text{ changes to } \frac{Y(x+\Delta x) - Y(x)}{\Delta x} = Y(x) \text{ with } Y(0) = 1. \tag{10}$$

This is Euler's method of approximation. $Y(x)$ approaches $y(x)$ as $\Delta x \to 0$.

Let me compute compound interest when 1 year is divided into 12 months, and then 365 days. The interest rate is 100% and you start with $Y(0) = \$1$. If you only get interest once, at the end of the year, then you have $Y(1) = \$2$.

If interest is added every month, you now get $\frac{1}{12}$ of 100% each time (12 times). So Y is multiplied each month by $1+\frac{1}{12}$. (The bank adds $\frac{1}{12}$ for every 1 you have.) Do this 12 times and the final value \$2 is improved to \$2.61:

After 12 months $$Y(1) = \left(1 + \frac{1}{12}\right)^{12} = \$2.61$$

Now add interest every day. $Y(0) = \$1$ is multiplied 365 times by $1+\frac{1}{365}$:

After 365 days $$Y(1) = \left(1 + \frac{1}{365}\right)^{365} = \$2.71 \textit{ (close to e)}$$

Very few banks use minutes, and nobody divides the year into $N = 31,536,000$ seconds. It would add less than a penny to \$2.71. But many banks are willing to use *continuous compounding*, the limit as $N \to \infty$. After one year you have \$$e$:

Another limit gives e $$\left(1 + \tfrac{1}{N}\right)^N \to e = 2.718\ldots \text{ as } N \to \infty \tag{11}$$

You could invest at the 100% rate for x years. Now each of the N steps is for x/N years. Again the bank multiplies at every step by $1+\frac{x}{N}$. The 1 keeps what you have, the x/N adds the interest in that step. After N steps you are close to e^x:

A formula for e^x $$\left(1 + \tfrac{x}{N}\right)^N \to e^x \text{ as } N \to \infty \tag{12}$$

Finally, I will change the interest rate to a. Go for x years at the interest rate a. The differential equation changes from $dy/dx = y$ to $\boldsymbol{dy/dx = ay}$. The exponential function still solves it, but now that solution is $y = e^{ax}$:

Change the rate to a $$\frac{dy}{dx} = ay \textit{ is solved by } y(x) = e^{ax} \tag{13}$$

You can write down the series $e^{ax} = 1 + ax + \frac{1}{2}(ax)^2 + \cdots$ and take its derivative:

$$\frac{d}{dx}(e^{ax}) = a + a^2x + \cdots = a(1 + ax + \cdots) = \boldsymbol{ae^{ax}} \tag{14}$$

The derivative of e^{ax} brings down the extra factor a. So $y = e^{ax}$ solves $dy/dx = ay$.

The Exponential $y = e^x$

Looking for a function $y(x)$ that equals its own derivative $\dfrac{dy}{dx}$

A differential equation! We start at $x = 0$ with $y = 1$

Infinite Series $y(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \left(\dfrac{x^n}{n!}\right) + \cdots$

Take derivative $\dfrac{dy}{dx} = 0 + 1 + x + \dfrac{x^2}{2!} + \cdots + \left(\dfrac{x^{n-1}}{(n-1)!}\right) + \cdots$

Term by term $\dfrac{dy}{dx}$ agrees with y Limit step = add up this series

$n! = (n)(n-1)\cdots(1)$ grows much faster than x^n so the terms get very small

At $x = 1$ the number $y(1)$ is called e. Set $x = 1$ in the series to find e

$e = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots = \mathbf{2.71828\ldots}$

GOAL Show that $y(x)$ agrees with e^x for all x Series gives powers of e

Check that the series follows the rule to add exponents as in $e^2e^3 = e^5$

Directly multiply series e^x times e^X to get e^{x+X}

$\left(1 + x + \dfrac{1}{2}x^2\right)$ times $\left(1 + X + \dfrac{1}{2}X^2\right)$ produces the right start for e^{x+X}

$1 + (x + X) + \dfrac{1}{2}(x + X)^2 + \cdots$ HIGHER TERMS ALSO WORK

The series gives us e^x for EVERY x, not just whole numbers

CHECK $\dfrac{de^x}{dx} = \lim \dfrac{e^{x+\Delta x} - e^x}{\Delta x} = e^x\left(\lim \dfrac{e^{\Delta x} - 1}{\Delta x}\right) = e^x$ YES!

SECOND KEY RULE $\boldsymbol{(e^x)^n = e^{nx}}$ for every x and n

Another approach to e^x uses multiplication instead of an infinite sum

Start with \$1. Earn interest every day at yearly rate x

Multiply 365 times by $\left(1 + \dfrac{x}{365}\right)$. End the year with \$ $\left(1 + \dfrac{x}{365}\right)^{365}$

Now pay n times in the year. End the year with $\left(1 + \dfrac{x}{n}\right)^n \to \$\, e^x$ as $n \to \infty$

We are solving $\dfrac{\Delta Y}{\Delta x} = Y$ in n short steps Δx. The limit solves $\dfrac{dy}{dx} = y$.

Practice Questions

1. What is the derivative of $\dfrac{x^{10}}{10!}$? What is the derivative of $\dfrac{x^9}{9!}$?
2. How to see that $\dfrac{x^n}{n!}$ gets small as $n \to \infty$?

Start with $\dfrac{x}{1}$ and $\dfrac{x^2}{2}$, possibly big. But we multiply by $\dfrac{x}{3}, \dfrac{x}{4}, \cdots$ which gets small.

3. Why is $\dfrac{1}{e^x}$ the same as e^{-x} ? Use equation (3) and also use (6).
4. Why is $e^{-1} = 1 - 1 + \dfrac{1}{2} - \dfrac{1}{6} + \cdots$ between $\dfrac{1}{3}$ and $\dfrac{1}{2}$? Then $2 < e < 3$.
5. Can you solve $\dfrac{dy}{dx} = y$ starting from $y = 3$ at $x = 0$?

Why is $y = 3e^x$ the right answer ? Notice how 3, multiplies e^x.

6. Can you solve $\dfrac{dy}{dx} = 5y$ starting from $y = 1$ at $x = 0$?

Why is $y = e^{5x}$ the right answer ? Notice 5 in the exponent!

7. Why does $\dfrac{e^{\Delta x} - 1}{\Delta x}$ approach 1 as Δx gets smaller ? Use the $e^{\Delta x}$ series.

8. Draw the graph of $x = \ln y$, just by flipping the graph of $y = e^x$ across the 45° line $y = x$. Remember that y stays positive but $x = \ln y$ can be negative.
9. What is the exact sum of $1 + \ln 2 + \dfrac{1}{2}(\ln 2)^2 + \dfrac{1}{3!}(\ln 2)^3 + \cdots$?
10. If you replace ln 2 by 0.7, what is the sum of those four terms ?
11. From Euler's Great Formula $e^{ix} = \cos x + i \sin x$, what number is $e^{2\pi i}$?
12. How close is $\left(1 + \dfrac{1}{10}\right)^{10}$ to e ?
13. What is the limit of $\left(1 + \dfrac{1}{N}\right)^{2N}$ as $N \to \infty$?