Math Lab 6: Implicit Differentiation and the Mean Value Theorem

Goals: Practice using the chain rule as applied to implicit differentiation, and apply implicit differentiation to problems. Explore the Mean Value Theorem.

Groups: Work by yourself (feel free to consult with any classmates of your choice) or with a partner of your choice. If working with a partner, for any work involving Desmos, make sure both partners know what to do (your individual ability to use Desmos is important).

Lab Notebook: Standard Lab Notebook guidelines. See earlier lab handouts for details. Use your best judgement about what graphs to include. Recall that for math and physics labs, your notebook should reflect your learning and your engagement, and should serve as a stand-alone representation of what you did and what you learned. **Note: Lab Notebook Check – turn in math & physics lab notebooks outside Krishna's office Lab 2 3255 by 5 pm Friday November 20.**

Part 0. Program Website Check-In (complete before 10:15 or do after class)

1. Go to our program website sites.evergreen.edu/mnm1516. Remind yourself of the resources available under Math Resources. Note that comments are turned on at that page, so feel free to suggest math resources not already suggested that you find useful and think might be of use to your fellow students (do this after class).
2. Find the Solutions page. Consider how you have been using the provided solutions. What are some ways you would like to use those solutions better, OR, what advice would you give to students who would like to use the solutions better? What are some ways that the faculty can help you to make better use of the solutions?
3. Read the post on Math and Physics Lab Notebook Check.

Part 1. Implicit Differentiation (work on this until 11:15. All of these are homework problems, so you can use the below later to help you get you started on these or check your work)

1. We begin as a class by reviewing implicitly defined functions ("implicit functions") and how to use the chain rule to find their derivatives ("implicit differentiation"). We will work together on 3.7.31 (from your Reading Response): given $x^2 + y^2 - 4x + 7y = 15$, find $y' = \frac{dy}{dx}$. Under what conditions on x and/or y is the tangent line to this curve horizontal? Vertical?
2. Launch Desmos and plot $x^2 + y^2 - 4x + 7y = 15$ just by typing it in directly. Does your plot confirm what you found about horizontal and vertical tangents?
3. Consider
$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$
where x_0, y_0, a, b are constants. Do you know what shape this is? Plot it in Desmos, using sliders for the constants. Change the sliders and see what happens to the shape. How do you make a circle centered at the origin? How do you change the radius of the circle? How do you change the location of the circle's center? What about an ellipse with a horizontal axis longer than its vertical axis? Vertical longer than horizontal? *(Do this later: how do you make a tilted ellipse, whose axis are not parallel to the x- and y- axis? You might need to adjust more than just slider values.)*
4. (based on 3.7.33) Don't use Desmos until directed. Consider
$$\frac{(x - 2)^2}{16} + \frac{y^2}{4} = 1$$
Based on your work above, can you guess the shape and any details of this implicit function? Now, plot it in Desmos to check your guess. Find $y' = \frac{dy}{dx}$ and determine if there are any horizontal or vertical tangents; see if they match your plot. Next, find equations of the tangent lines at $x = 0$. Plot those lines on your graph to confirm.
5. (based on 3.7.23) Find the equation of the tangent line to the curve $\sin(xy) = x$ at $(1, \frac{\pi}{2})$. Then, plot the curve and the tangent line to confirm. If you're not sure that you've found the correct y', discuss with your partner, a neighbor, or an instructor.
6. (based on 3.7.36) Find the equation of the tangent line to the curve $y = x^2$ at $x = 1$ (note: this is a standarad problem you could have done before studying implicit differentiation). Plot the curve and the tangent line to confirm. Also, plot a circle centered at $(8, 0)$ with arbitrary radius R: its equation would be $(x - 8)^2 + y^2 = R^2$ (with a slider for R) – does that make sense to you? Show by adjusting R that the tangent line to $y = x^2$ at $x = 1$ is also tangent to some circle centered at $(8, 0)$. Note the value for R that you found by this graphical approach. Now, design an approach using calculus that finds the exact value for R.
7. (based on 3.7.39) If $y = \arcsin x$, then $x = \sin y$. Use implicit differentiation (and some trig definitions or trig identities) on $x = \sin y$ to show that
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

Part 2. Logarithmic Differentiation (11:15 – 11:30)

1. Consider the function $y = x^x$. Without plotting it, do you have a guess for its shape? Its domain (e.g. any x values that are forbidden)? Its range (any limits on y-values)?
2. Now, plot it in Desmos. Do you note that its range is limited to $x \geq 0$? Do you notice any horizontal tangents? Do you see that the horizontal tangent is associated with the minimum value of the function?
3. Can you find y' using the differentiation rules we have developed so far? Will the Power Rule work? Will the Exponential Rule work? Try them and see what you get or what difficulties you run into. How do you know that what you found doesn't work? (Hint: do you get a horizontal tangent at the right location? Does your derivative function match slopes of the original function?)
4. Since $y = x^x$ is neither a Power Function (the power isn't constant) nor an Exponential Function (the base isn't constant), we have to use a different approach. Instead, try this: take the natural logarithm of both sides of $y = x^x$. Look up some logarithm rules, and show that you get $\ln y = x \ln x$.
5. Now, use implicit differentiation to find y'. Normally, we leave this in terms of x and y because we can't easily get y in terms of x, but here, we know that $y = x^x$. So write down y' in terms of x.
6. Set this equal to zero and solve for x. Does this match the location of the horizontal tangent of the original function? Plot your y' and see if the derivative matches the slopes of the original function.
7. Finish off by finding the exact location and value of the minimum of $y = x^x$.
8. Bring your questions to calculus lecture.

Part 3. The Mean Value Theorem (11:30 – 11:45)

1. If you have time and know-how, produce a Desmos calculator that does the following. Otherwise, just use the one I developed (**https://www.desmos.com/calculator/bignnxibvy**). Produce a Desmos calculator that plots some arbitrary function $f(x)$. It should also plot the secant line through some arbitrary points $(a, f(a))$ and $(b, f(b))$ (the points should be movable points, and you should keep $a < b$) and also output the slope m of the secant line. It should also plot the tangent line to the function at some moveable arbitrary point $x = c$, where you will keep $a < c < b$, as well as output the slope of the tangent line (i.e. the derivative $f'(a)$).
2. Try a straightforward function, like $f(x) = x^2$. Set an a and b so that the secant line is not horizontal. Adjust your c so that the slope of the tangent line at $x = c$ matches the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$.
3. Now, try a more complicated $f(x)$, such as $f(x) = xe^x$ or x^2e^{-x} or $x \sin x$ or your choice, and repeat the above, adjusting c.
4. Now, try $f(x) = |x|$. What happens? Can you find a c such that the slope of the tangent line matches the slope of the secant line for any secant line?
5. Bring your questions to calculus lecture.